

Affine symmetry in mechanics of collective and internal modes. Part II. Quantum models

J. J. Sławianowski, V. Kovalchuk, A. Sławianowska,
 B. Gołubowska, A. Martens, E. E. Rożko, Z. J. Zawistowski
 Institute of Fundamental Technological Research,
 Polish Academy of Sciences,
 21 Świętokrzyska str., 00-049 Warsaw, Poland
 e-mails: jslawian@ippt.gov.pl, vkoval@ippt.gov.pl,
 aslawian@ippt.gov.pl, bgolub@ippt.gov.pl,
 amartens@ippt.gov.pl, erozko@ippt.gov.pl,
 zzawist@ippt.gov.pl

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Abstract

Discussed is the quantized version of the classical description of collective and internal affine modes as developed in Part I. We perform the Schrödinger quantization and reduce effectively the quantized problem from n^2 to n degrees of freedom. Some possible applications in nuclear physics and other quantum many-body problems are suggested. Discussed is also the possibility of half-integer angular momentum in composed systems of spin-less particles.

Keywords: collective modes, affine invariance, Schrödinger quantization, quantum many-body problem.

Introduction

A fascinating feature of our models of affine collective dynamics is their extremely wide range of applications. It covers the nuclear and molecular dynamics, micromechanics of structured continua, perhaps nanostructure and defects phenomena, macroscopic elasticity and astrophysical phenomena like vibration of stars and clouds of cosmic dust. Obviously, microphysical applications must be based on the quantized version of the theory. And one is dealing then with a very curious convolution of quantum theory with mathematical methods of continuum mechanics. It is worth to mention that there were even attempts, mainly by Barut and Rączka [4], to describe the dynamics of strongly interacting elementary particles (hadrons) in terms of some peculiar, quantized continua.

By the way, as French say, the extremes teach one another; it is not excluded that the dynamics of cosmic objects like neutron stars must be also described in quantum terms. They are though giant nuclei, very exotic ones, because composed exclusively of neutrons (enormous "mass numbers" and vanishing "atomic numbers").

1 Quantization of classical geodetic systems

As usual, before quantizing the classical model, one has to perform some preliminary work on the level of its classical Hamiltonian dynamics [11, 16, 17, 18, 19].

Let us consider a classical geodetic system in a Riemannian manifold (Q, Γ) , where Q denotes the configuration space, and Γ is the "metric" tensor field on Q underlying the kinetic energy form. In terms of generalized coordinates or in Hamiltonian terms we have, respectively,

$$T = \frac{1}{2} \Gamma_{\mu\nu} \frac{dq^\mu}{dt} \frac{dq^\nu}{dt}, \quad \mathcal{T} = \frac{1}{2} \Gamma^{\mu\nu} p_\mu p_\nu,$$

where, obviously, $\Gamma^{\mu\alpha} \Gamma_{\alpha\nu} = \delta^\mu_\nu$, $p_\mu = \partial T / \partial \dot{q}^\mu = \Gamma_{\mu\nu} (dq^\nu / dt)$.

As usual, the metric tensor Γ gives rise to the natural measure μ_Γ on Q ,

$$d\mu_\Gamma(q) = \sqrt{|\det[\Gamma_{\mu\nu}]|} dq^1 \cdots dq^f,$$

where f denotes the number of degrees of freedom, i.e., $f = \dim Q$. For simplicity the square-root expression will be always denoted by $\sqrt{|\Gamma|}$. The mathematical framework of Schrödinger quantization is based on $L^2(Q, \mu_\Gamma)$, i.e., the Hilbert space of complex-valued wave functions on Q square-integrable in the μ_Γ -sense. Their scalar product is given by the usual formula:

$$\langle \Psi_1 | \Psi_2 \rangle = \int \bar{\Psi}_1(q) \Psi_2(q) d\mu_\Gamma(q).$$

The classical kinetic energy expression is replaced by the operator $\mathbf{T} = -(\hbar^2/2)\Delta(\Gamma)$, where \hbar denotes the ("crossed") Planck constant, and $\Delta(\Gamma)$ is the Laplace-Beltrami operator corresponding to Γ , i.e.,

$$\Delta(\Gamma) = \frac{1}{\sqrt{|\Gamma|}} \sum_{\mu, \nu} \partial_\mu \sqrt{|\Gamma|} \Gamma^{\mu\nu} \partial_\nu = \Gamma^{\mu\nu} \nabla_\mu \nabla_\nu.$$

In the last expression ∇_μ denotes the Levi-Civita covariant differentiation in the Γ -sense. Therefore, the kinetic energy operator \mathbf{T} is formally obtained from the corresponding classical expression \mathcal{T} (kinetic Hamiltonian) by the substitution $p_\mu \mapsto \mathbf{p}_\mu = (\hbar/i)\nabla_\mu$.

If the problem is non-geodetic and some potential $V(q)$ is admitted, the corresponding Hamilton (energy) operator is given by $\mathbf{H} = \mathbf{T} + \mathbf{V}$, where the operator \mathbf{V} acts on wave functions simply multiplying them by V , i.e., $(\mathbf{V}\Psi)(q) = V(q)\Psi(q)$. This is the reason why very often one does not distinguish graphically between \mathbf{V} and V .

2 Problems concerning quantization

There are, obviously, many delicate problems concerning quantization which cannot be discussed here and, fortunately, do not interfere directly with the main subjects of our analysis. Nevertheless, we mention briefly some of them. Strictly speaking, wave functions are not scalars but complex densities of the weight $1/2$ so that the bilinear expression $\bar{\Psi}\Psi$ is a real scalar density of weight one, thus, a proper object for describing probability distributions [10]. But in all realistic models, and the our one is not an exception, the configuration space is endowed with some Riemannian structure. And this enables one to factorize scalar (and tensor) densities into products of scalars (tensors) and some standard densities built of the metric tensor. Therefore, the wave function may be finally identified with the complex scalar field (multicomponent one when there are internal degrees of freedom).

There are also some arguments for modifying \mathbf{T} by some scalar term proportional to the curvature scalar. Of course, such a term may be always formally interpreted as some correction potential. And besides, we usually deal with Riemannian manifolds of the constant Riemannian curvature, and then such additional terms result merely in the over-all shifting of energy levels.

In Riemann manifolds the Levi-Civita affine connection preserves the scalar product; because of this, the operator ∇_μ is formally anti-self-adjoint and $(\hbar/i)\nabla_\mu$, $\mathbf{T} = -(\hbar^2/2)\Gamma^{\mu\nu}\nabla_\mu\nabla_\nu$ are formally self-adjoint. They are, however, differential operators, thus, the difficult problem of self-adjoint extensions appears. And besides, being differential operators, they are unbounded in the usual sense, thus, their spectral analysis also becomes a difficult and delicate subject. All such problems will be neglected and considered in the zeroth-order approximation of the mathematical rigor, just as it is usually done in practical physical applications. This is also justified by the fact that, as a rule, our first-order differential operators generate some well-definite global transformation groups admitting a lucid geometrical interpretation. It is typical that in such situation all subtle problems on the level of functional analysis, like the common domains, etc., may be successfully solved.

Therefore, from now on we will proceed in a "physical" way and all terms like "self-adjoint", "Hermitian", etc. will be used in a rough way characteristic for physical papers and applied mathematics.

We shall deal almost exclusively with stationary problems when the Hamilton operator \mathbf{H} is time-independent, thus, the Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t} = \mathbf{H}\psi$$

will be replaced by its stationary form, i.e., by the eigenequation $\mathbf{H}\Psi = E\Psi$, where, obviously,

$$\psi = \exp\left(-\frac{i}{\hbar}Et\right)\Psi$$

and Ψ is a time-independent wave function on the configuration space.

3 Multi-valuedness of wave functions

There is another delicate point concerning fundamental aspects of quantization which, however, may be of some importance and will be analyzed later on. Namely, it is claimed in all textbooks in quantum mechanics that wave functions solving reasonable Schrödinger equations must satisfy strong regularity conditions, and first of all they must be well-defined one-valued functions all over the configuration space, in addition, continuous together with their derivatives. This demand is mathematically essential in the theory of Sturm-Liouville equations and besides it has to do with quantization or, more precisely, discrete spectra of certain physical quantities. By the way, these two things are not independent.

There are, however, certain arguments that some physical systems may admit multi-valued wave functions. It is so when the configuration space is not simply connected and its fundamental group is finite. Physically it is only the squared modulus $\overline{\Psi}\Psi$ that is to be one-valued because, according to the Born statistical interpretation, it represents the probability distribution of detecting a system in various regions of the configuration space. But for the wave function Ψ itself it is sufficient to be "locally" one-valued and sufficiently smooth, i.e., to be defined on the universal covering manifold \overline{Q} of the configuration space Q . This may lead to a consistent quantum mechanics, perhaps with some kind of superselection rules. It is so in quantum mechanics of rigid body, which is sometimes expected to be a good model of the elementary particles spin [1, 2, 3]. The configuration space of the rigid body without translational motion may be identified with the proper rotation group $SO(3, \mathbb{R})$ ($SO(n, \mathbb{R})$ in n dimensions), obviously, when some reference orientation and Cartesian coordinates are fixed. But it is well-known that $SO(3, \mathbb{R})$ is doubly-connected (and so is $SO(n, \mathbb{R})$ for any $n \geq 3$). Its covering group is $SU(2)$ ($Spin(n)$ for any $n \geq 3$). Therefore, it is really an instructive exercise, and perhaps also a promising physical hypothesis, to develop the rigid top theory with $SU(2)$ as configuration space [1, 2, 3]. In affinely-rigid body mechanics we are dealing with a similar situation, namely, $GL(3, \mathbb{R})$ and $SL(3, \mathbb{R})$ (more generally, $GL(n, \mathbb{R})$ and $SL(n, \mathbb{R})$ for $n > 3$) are doubly-connected. This topological property is simply inherited from the corresponding one for $SO(3, \mathbb{R})$ ($SO(n, \mathbb{R})$) on the basis of the polar decomposition [4, 24, 25]. Therefore, the standard quantization procedure in a manifold should be modified by using wave amplitudes defined on the covering manifolds $\overline{GL(n, \mathbb{R})}$, $\overline{SL(n, \mathbb{R})}$. By the way, some difficulty and mathematical curiosity appears then because these covering groups are non-linear (do not admit faithful realizations in terms of finite-dimensional matrices). This fact, known long ago to E. Cartan, was not known to physicists; a rather long time and enormous work has been lost because of this.

4 Classical background for quantization

Before going into such details we must go back to certain classical structures underlying quantization procedure. They were touched earlier in sections 2 and 3 of Part I [20] but in a rather superficial way, and besides, we concentrated there on the collective modes ruled by the linear and affine groups. This is really the main objective of our study, nevertheless, not exceptional one; it is also clear that, injecting the subject into a wider context, one attains a deeper understanding, free of accidental details.

In section 2 of Part I [20] Lie-algebraic objects $\Omega, \hat{\Omega} \in G'$ were introduced. It is an important fact from the Lie group theory that they give rise to some vector fields X, Y on G invariant, respectively, under right and left translations on G . Namely, for any fixed $\Omega, \hat{\Omega} \in G'$, they are given by $X_g[\Omega] := \Omega g, Y_g[\hat{\Omega}] := g\hat{\Omega}$.

Affine velocities introduced in section 3 of Part I [20] are just the special case of Lie-algebraic objects. In the same section the dual objects $\Sigma, \hat{\Sigma}$, i.e., affine spin in two representations, were introduced. These dual quantities exist also in the general case when G is an arbitrary Lie group. They are then elements of the dual space, i.e., Lie co-algebra, $\Sigma, \hat{\Sigma} \in G'^*$. Their relationship with canonical momenta p and configurations g is given by the following formula involving evaluations of co-vectors on vectors: $\langle p, \dot{g} \rangle = \langle \Sigma, \Omega \rangle = \langle \hat{\Sigma}, \hat{\Omega} \rangle$, where $\dot{g} \in T_g G, p \in T_g^* G$, and g, \dot{g} are arbitrary. Denoting the adjoint transformation of Ad_g by the usual symbol Ad_g^* , we have that $\Sigma = \text{Ad}_g^{*-1} \hat{\Sigma}$, the obvious generalization of the corresponding relationship between laboratory and co-moving representation of affine (or usual metrical) spin. And just as in this special case, the quantities $\Sigma, \hat{\Sigma}$ are Hamiltonian generators of the groups of left and right regular translations L_G, R_G on G .

In applications we are usually dealing with some special Lie groups for which many important formulas and relationships may be written in a technically simple form avoiding the general abstract terms.

As mentioned, throughout this series of articles we are dealing almost exclusively with linear groups $G \in \text{GL}(W) \subset \text{L}(W)$, where W is a linear space, e.g., some \mathbb{R}^n or \mathbb{C}^n .

All the mentioned simplifications follow from the obvious canonical isomorphism between $\text{L}(W)$ and its dual $\text{L}(W)^*$, based on the pairing $\langle C, D \rangle = \text{Tr}(CD)$. The Lie algebra G' is a linear subspace of $\text{L}(W)$, therefore, its dual space G'^* may be canonically identified with the quotient space $\text{L}(W)^*/\text{An}G'$, where $\text{An}G'$ denotes the subspace of linear functions vanishing on G' . But, according to the above identification between $\text{L}(W)^*$ and $\text{L}(W)$ itself, $\text{An}G'$ may be identified with some linear subspace of $\text{L}(W)$; we shall denote it by G'^\perp . Therefore, the Lie co-algebra G'^* is canonically isomorphic with the corresponding quotient, i.e., $G'^* \simeq \text{L}(W)/G'^\perp$. This is the general fact for linear groups and their Lie algebras. However, in some special cases, just ones of physical relevance, this quotient space admits a natural canonical isomorphism onto some distinguished linear subspace of $\text{L}(W)$ consisting of natural representants of cosets, e.g., in the most practical cases G'^* is canonically iso-

morphic with G' itself. For example, it is so for $\mathrm{SO}(n, \mathbb{R})$, $\mathrm{SL}(n, \mathbb{R})$, where the Lie algebras $\mathrm{SO}(n, \mathbb{R})'$, $\mathrm{SL}(n, \mathbb{R})'$ may be identified with the duals $\mathrm{SO}(n, \mathbb{R})'^*$, $\mathrm{SL}(n, \mathbb{R})'^*$. By the way, for certain reasons it is more convenient to use the pairing $\langle A, B \rangle = -(1/2)\mathrm{Tr}(AB)$ for the orthogonal group $\mathrm{SO}(n, \mathbb{R})$.

Just as in the special case of affine objects, transformation rules for Σ , $\hat{\Sigma}$ are analogous to those for Ω , $\hat{\Omega}$; we mean transformations under regular translations:

$$\begin{aligned} L_k &: \quad \Sigma \mapsto \mathrm{Ad}_k^{*-1} \Sigma, & \hat{\Sigma} &\mapsto \hat{\Sigma}, \\ R_k &: \quad \Sigma \mapsto \Sigma, & \hat{\Sigma} &\mapsto \mathrm{Ad}_k^* \hat{\Sigma}. \end{aligned}$$

Using the identifications mentioned above (assuming that they work), we can write these rules in a form analogous to that for non-holonomic velocities,

$$\begin{aligned} L_k &: \quad \Sigma \mapsto k \Sigma k^{-1}, & \hat{\Sigma} &\mapsto \hat{\Sigma}, \\ R_k &: \quad \Sigma \mapsto \Sigma, & \hat{\Sigma} &\mapsto k^{-1} \hat{\Sigma} k, \end{aligned}$$

i.e., just as it is for the affine spin.

Geometrical meaning of Σ and $\hat{\Sigma}$ is that of the momentum mappings induced, respectively, by the group of left and right regular translations. And the relationship between two versions of Σ -objects is as follows: $\Sigma = g \hat{\Sigma} g^{-1}$. The objects Σ and $\hat{\Sigma}$ may be also interpreted in terms of right- and left-invariant differential forms (co-vector fields), i.e., Maurer-Cartan forms A, B on the group G . Assuming the afore-mentioned identification, we can express A, B for any fixed $\Sigma, \hat{\Sigma}$ in the following forms: $A_g[\Sigma] = g^{-1} \Sigma$, $B_g[\hat{\Sigma}] = \hat{\Sigma}^{-1} g$.

Just as in the special case of affine systems, Poisson bracket relations of Σ - and $\hat{\Sigma}$ -components are given by structure constants of G . Those for $\hat{\Sigma}$ have opposite signs to those for Σ , and the mutual ones vanish (left regular translations commute with the right ones).

5 Hamiltonian systems on Lie group spaces

Geodetic Hamiltonian systems on Lie group spaces were studied by various research groups; let us mention, e.g., the prominent mathematicians like Hermann, Arnold, Mishchenko, Fomenko, and others. Obviously, the special stress was laid on models with kinetic energies (Riemann structures on G) invariant under left or right regular translations. As expected, models invariant simultaneously under left and right translations have some special properties and due to their high symmetries are computationally simplest.

From now on we assume that our configuration space Q is a Lie group G or, more precisely, its homogeneous space with trivial isotropy groups. Also in a more general situation when isotropy groups are nontrivial (even continuous) a large amount of analysis performed on group spaces remains useful.

Obviously, just as in the special case of affinely-rigid bodies, left- and right-invariant kinetic energies T are, respectively, quadratic forms of $\hat{\Omega}$ and Ω with constant coefficients. Their underlying Riemannian structures on G are locally flat if and only if G is Abelian.

In both theoretical and practical problems the Hamilton language based on Poisson brackets is much more lucid and efficient than that based on Lagrange equations. If besides of geodetic inertia the system is influenced only by potential forces derivable from some potential energy term $V(q)$, then, obviously, the classical Hamiltonian is given by the following expression:

$$H = \mathcal{T} + V(q) = \frac{1}{2} \Gamma^{\mu\nu}(q) p_\mu p_\nu + V(q).$$

It is very convenient to express the Hamiltonian and all other essential quantities in terms of non-holonomic velocities and their conjugate non-holonomic (Poisson-non-commuting) momenta.

Let $\{E_\mu\}$ be some basis in the Lie algebra G' and q^μ be the corresponding canonical coordinates of the first kind on G , i.e., $g(q) = \exp(q^\mu E_\mu)$. Lie-algebraic objects $\Omega, \hat{\Omega} \in G'$ will be, respectively, expanded as follows: $\Omega = \Omega^\mu E_\mu$, $\hat{\Omega} = \hat{\Omega}^\mu E_\mu$. Using the expansion coefficients Ω^μ , $\hat{\Omega}^\mu$ one obtains the following simple expressions for the left- and right-invariant kinetic energies:

$$T_{\text{left}} = \frac{1}{2} \mathcal{L}_{\mu\nu} \hat{\Omega}^\mu \hat{\Omega}^\nu, \quad T_{\text{right}} = \frac{1}{2} \mathcal{R}_{\mu\nu} \Omega^\mu \Omega^\nu,$$

where the matrices \mathcal{L} , \mathcal{R} are constant, symmetric, and non-singular. The positive definiteness problem is a more delicate matter, and there are some hyperbolic-signature structures of some relevance both for physics and pure geometry.

For potential systems Legendre transformation may be easily described with the use of non-holonomic objects, respectively,

$$\hat{\Sigma}_\mu = \frac{\partial T_{\text{left}}}{\partial \hat{\Omega}^\mu} = \mathcal{L}_{\mu\nu} \hat{\Omega}^\nu, \quad \Sigma_\mu = \frac{\partial T_{\text{right}}}{\partial \Omega^\mu} = \mathcal{R}_{\mu\nu} \Omega^\nu,$$

where, obviously, $\hat{\Sigma}_\mu$, Σ_μ are expansion coefficients of $\hat{\Sigma}$, Σ with respect to the dual basis $\{E^\mu\}$ of the Lie co-algebra, i.e., $\hat{\Sigma} = \hat{\Sigma}_\mu E^\mu$, $\Sigma = \Sigma_\mu E^\mu$. The resulting Hamiltonians have, respectively, the following forms:

$$H = \mathcal{T}_{\text{left}} + V(q) = \frac{1}{2} \mathcal{L}^{\mu\nu} \hat{\Sigma}_\mu \hat{\Sigma}_\nu + V(q),$$

$$H = \mathcal{T}_{\text{right}} + V(q) = \frac{1}{2} \mathcal{R}^{\mu\nu} \Sigma_\mu \Sigma_\nu + V(q),$$

where, obviously, the matrices $[\mathcal{L}^{\mu\nu}]$, $[\mathcal{R}^{\mu\nu}]$ are reciprocal to $[\mathcal{L}_{\mu\nu}]$, $[\mathcal{R}_{\mu\nu}]$.

If structure constants of G' with respect to the basis $\{E_\mu\}$ are defined according to the convention $[E_\mu, E_\nu] = E_\lambda C^\lambda_{\mu\nu}$, then the Poisson brackets of Σ -objects are given as follows:

$$\{\Sigma_\mu, \Sigma_\nu\} = \Sigma_\lambda C^\lambda_{\mu\nu}, \quad \{\hat{\Sigma}_\mu, \hat{\Sigma}_\nu\} = -\hat{\Sigma}_\lambda C^\lambda_{\mu\nu}, \quad \{\Sigma_\mu, \hat{\Sigma}_\nu\} = 0.$$

6 Basic differential operators

Let us define basic differential operators generating left and right regular translations on G . We denote them respectively by \mathbf{L}_μ and \mathbf{R}_μ . Their action on complex- or vector-valued functions F on G is defined as follows:

$$(\mathbf{L}_\mu F)(g) := \left. \frac{\partial}{\partial q^\mu} F(k(q)g) \right|_{q=0}, \quad (\mathbf{R}_\mu F)(g) := \left. \frac{\partial}{\partial q^\mu} F(gk(q)) \right|_{q=0}. \quad (1)$$

Their Lie-bracket (commutator) relations differ from the above Poisson rules for Σ -quantities by signs:

$$[\mathbf{L}_\mu, \mathbf{L}_\nu] = -\mathbf{L}_\lambda C^\lambda_{\mu\nu}, \quad [\mathbf{R}_\mu, \mathbf{R}_\nu] = \mathbf{R}_\lambda C^\lambda_{\mu\nu}, \quad [\mathbf{L}_\mu, \mathbf{R}_\nu] = 0.$$

Poisson brackets between Σ -objects and functions F depending only on coordinates q (pull-backs of functions defined on the configuration space $Q = G$) are given by

$$\{\Sigma_\mu, F\} = -\mathbf{L}_\mu F, \quad \{\hat{\Sigma}_\mu, F\} = -\mathbf{R}_\mu F.$$

The system of Poisson brackets quoted above is sufficient for calculating any other Poisson bracket with the help of well-known properties of this operation. Thus, e.g., for any pair of functions A, B depending in general on all phase-space variables we have the following expression:

$$\{A, B\} = \Sigma_\lambda C^\lambda_{\mu\nu} \frac{\partial A}{\partial \Sigma_\mu} \frac{\partial B}{\partial \Sigma_\nu} - \frac{\partial A}{\partial \Sigma_\mu} \mathbf{L}_\mu B + \frac{\partial B}{\partial \Sigma_\mu} \mathbf{L}_\mu A,$$

and, when the phase space is parameterized in terms of quantities $q^\mu, \hat{\Sigma}_\mu$, we have the similar expression:

$$\{A, B\} = -\hat{\Sigma}_\lambda C^\lambda_{\mu\nu} \frac{\partial A}{\partial \hat{\Sigma}_\mu} \frac{\partial B}{\partial \hat{\Sigma}_\nu} - \frac{\partial A}{\partial \hat{\Sigma}_\mu} \mathbf{R}_\mu B + \frac{\partial B}{\partial \hat{\Sigma}_\mu} \mathbf{R}_\mu A.$$

Obviously, the finite regular translations may be expressed in terms of the following exponential formulas:

$$F(k(q)g) = \exp(q^\mu \mathbf{L}_\mu) F, \quad F(gk(q)) = \exp(q^\mu \mathbf{R}_\mu) F, \quad (2)$$

with all known provisos concerning exponentiation of differential operators.

Non-holonomic velocities $\Omega, \hat{\Omega}$ depend linearly on generalized velocities \dot{q} , i.e., $\Omega^\mu = \Omega^\mu_\nu(q) \dot{q}^\nu$, $\hat{\Omega}^\mu = \hat{\Omega}^\mu_\nu(q) \dot{q}^\nu$. Similarly, Σ and $\hat{\Sigma}$ depend contragradiently on the conjugate momenta p , i.e., $\Sigma_\mu = p_\alpha \Sigma^\alpha_\mu(q)$, $\hat{\Sigma}_\mu = p_\alpha \hat{\Sigma}^\alpha_\mu(q)$, where, obviously, $\Sigma^\alpha_\mu \Omega^\mu_\beta = \delta^\alpha_\beta$, $\hat{\Sigma}^\alpha_\mu \hat{\Omega}^\mu_\beta = \delta^\alpha_\beta$. This leads to the following expressions for generators:

$$\mathbf{L}_\mu = \Sigma^\alpha_\mu \frac{\partial}{\partial q^\alpha}, \quad \mathbf{R}_\mu = \hat{\Sigma}^\alpha_\mu \frac{\partial}{\partial q^\alpha}.$$

Many of the above statements remain true for the general non-holonomic velocities and their conjugate momenta without group-theoretical background

[5]. Nevertheless, there are also important facts depending on the group structure and on the properties of Σ_μ , $\hat{\Sigma}_\mu$ respectively as the basic right- and left-invariant co-vector fields (Maurer-Cartan forms). This concerns mainly invariant volumes, scalar products, Hermiticity of basic operators, and structure of the Laplace-Beltrami operator.

In group manifolds we are usually interested in left- or right-invariant kinetic energies. Even in the special case of the double invariance the definition-based direct calculation of the corresponding Laplace-Beltrami operator and the volume element may be rather complicated. However, if the corresponding kinetic metrics is left- or right-invariant, then so is the resulting volume element. Therefore, the L^2 -structure on G may be directly based on the integration with respect to the Haar measure. As known from the theory of locally compact groups, this measure is unique up to the constant normalization factor. In the special case of compact groups this normalization may be fixed by the natural demand that the total (finite in this case) volume equals to unity. In any case, the normalization is non-essential. In applications one deals usually with so-called unimodular groups, where the left and right measures coincide [9, 13]. Obviously, for the left- or right-invariant kinetic energies the measures μ_Γ built of the underlying metrics Γ are also left- or right-invariant. Therefore, they coincide with the Haar measure. This enables one to use the Haar measure from the very beginning as the integration prescription underlying the scalar product definition. This is very convenient for two reasons. First of all, for typical Lie groups appearing in physical applications the Haar measures are explicitly known. Another nice and reasonable feature of such a procedure is that once fixing the normalization we are given the standard integration procedure, whereas the use of $d\mu_\Gamma = \sqrt{|\Gamma|} dq^1 \cdots dq^f$ changes the scalar product normalization for various models of T (of Γ). This constant factor change is not very essential, but its dependence on various inertial parameters like the above I , A , B obscures the comparison of various models.

7 Unitary transformations

It follows from the very nature of the Haar measure μ that on the level of wave functions the left and right regular translations are realized by unitary transformations on $L^2(G, \mu)$. More precisely, let us define for any $k \in G$ the operators $\mathbf{L}(k)$, $\mathbf{R}(k)$ given by $(\mathbf{L}(k)\Psi)(g) := \Psi(kg)$, $(\mathbf{R}(k)\Psi)(g) := \Psi(gk)$ for any $g \in G$. It is clear that $\mathbf{L}(k)$, $\mathbf{R}(k)$ preserve the space $L^2(G, \mu)$, moreover, they are unitary transformations,

$$\langle \mathbf{L}(k)\Psi_1 | \mathbf{L}(k)\Psi_2 \rangle = \langle \mathbf{R}(k)\Psi_1 | \mathbf{R}(k)\Psi_2 \rangle = \langle \Psi_1 | \Psi_2 \rangle.$$

The assignments $G \ni k \mapsto \mathbf{L}(k)$, $\mathbf{R}(k)$ are, respectively, a unitary anti-representation and representation of G in $L^2(G, \mu)$, i.e.,

$$\mathbf{L}(k_1 k_2) = \mathbf{L}(k_2) \mathbf{L}(k_1), \quad \mathbf{R}(k_1 k_2) = \mathbf{R}(k_1) \mathbf{R}(k_2).$$

To convert \mathbf{L} into representation it is sufficient to replace $\Psi(kg)$ by $\Psi(k^{-1}g)$. Obviously, the difference is rather cosmetical and related to the conventions concerning the definition of the superposition of mappings. Nevertheless, any neglect may lead to the accumulation of sign errors and finally to numerically wrong results.

The operators $\mathbf{L}_\mu, \mathbf{R}_\mu$ generate the above representations, thus, we have

$$\mathbf{L}(\exp(q^\mu E_\mu)) = \exp(q^\mu \mathbf{L}_\mu), \quad \mathbf{R}(\exp(q^\mu E_\mu)) = \exp(q^\mu \mathbf{R}_\mu),$$

with all known provisos concerning domains and exponents of evidently unbounded differential operators. It is important to remember that the left-hand sides are always well-defined bounded unitary operators acting on the whole $L^2(G, \mu)$. Unlike this, $\mathbf{L}_\mu, \mathbf{R}_\mu$ act only on differentiable functions, they are unbounded, and the problems of domain and convergence appear on the right-hand sides of the above equations.

Unitarity of \mathbf{L}, \mathbf{R} implies that their generators $\mathbf{L}_\mu, \mathbf{R}_\mu$ are formally anti-self-adjoint (physicists tell roughly: anti-Hermitian), i.e.,

$$\langle \mathbf{L}_\mu \Psi_1 | \Psi_2 \rangle = -\langle \Psi_1 | \mathbf{L}_\mu \Psi_2 \rangle, \quad \langle \mathbf{R}_\mu \Psi_1 | \Psi_2 \rangle = -\langle \Psi_1 | \mathbf{R}_\mu \Psi_2 \rangle,$$

assuming that the left- and right-hand sides are well-defined (this is the case, e.g., for differentiable compactly supported functions on G).

Now, let us introduce the following operators:

$$\Sigma_\mu := \frac{\hbar}{i} \mathbf{L}_\mu, \quad \hat{\Sigma}_\mu := \frac{\hbar}{i} \mathbf{R}_\mu. \quad (3)$$

They are formally self-adjoint, i.e., "Hermitian" in the rough language of quantum physicists:

$$\langle \Sigma_\mu \Psi_1 | \Psi_2 \rangle = \langle \Psi_1 | \Sigma_\mu \Psi_2 \rangle, \quad \langle \hat{\Sigma}_\mu \Psi_1 | \Psi_2 \rangle = \langle \Psi_1 | \hat{\Sigma}_\mu \Psi_2 \rangle,$$

with the same as previously provisos concerning the functions Ψ_1, Ψ_2 . Obviously, \hbar denotes the ("crossed") Planck constant.

The operators $\Sigma_\mu, \hat{\Sigma}_\mu$ are quantized counterparts of classical physical quantities $\Sigma_\mu, \hat{\Sigma}_\mu$. They may be expressed as follows:

$$\Sigma_\mu = \frac{\hbar}{i} \Sigma^\alpha{}_\mu(q) \frac{\partial}{\partial q^\alpha}, \quad \hat{\Sigma}_\mu = \frac{\hbar}{i} \hat{\Sigma}^\alpha{}_\mu(q) \frac{\partial}{\partial q^\alpha}.$$

There is no problem of ordering of q -variables and differential operators $\partial/\partial q^\alpha$. This ordering is exactly as above, just due to the interpretation of \mathbf{L}_μ and \mathbf{R}_μ as infinitesimal generators of one-parameter subgroups.

8 Quantum Poisson bracket

In virtue of the above group-theoretical arguments the quantum Poisson-bracket rules are analogous to the classical ones,

$$_Q\{\Sigma_\mu, \Sigma_\nu\} = \Sigma_\lambda C^\lambda{}_{\mu\nu}, \quad _Q\{\hat{\Sigma}_\mu, \hat{\Sigma}_\nu\} = -\hat{\Sigma}_\lambda C^\lambda{}_{\mu\nu}, \quad _Q\{\Sigma_\mu, \hat{\Sigma}_\nu\} = 0.$$

Let us remind that the quantum Poisson bracket of operators is defined as

$$_Q\{\mathbf{A}, \mathbf{B}\} := \frac{1}{i\hbar}[\mathbf{A}, \mathbf{B}] = \frac{1}{i\hbar}(\mathbf{AB} - \mathbf{BA}).$$

One can show (see, e.g., [5]) that the kinetic energy operators for the left- and right-invariant models are given simply by the formerly quoted formulas with the classical generators Σ_μ , $\hat{\Sigma}_\mu$ replaced by the corresponding operators Σ_μ , $\hat{\Sigma}_\mu$, i.e.,

$$\begin{aligned} \mathcal{T}_{\text{left}} &= \frac{1}{2}\mathcal{R}^{\mu\nu}\hat{\Sigma}_\mu\hat{\Sigma}_\nu = -\frac{\hbar^2}{2}\mathcal{R}^{\mu\nu}\mathbf{R}_\mu\mathbf{R}_\nu, \\ \mathcal{T}_{\text{right}} &= \frac{1}{2}\mathcal{L}^{\mu\nu}\Sigma_\mu\Sigma_\nu = -\frac{\hbar^2}{2}\mathcal{L}^{\mu\nu}\mathbf{L}_\mu\mathbf{L}_\nu. \end{aligned}$$

As mentioned, the literal calculation of the Laplace-Beltrami operator in terms of local coordinates q^μ is usually very complicated and the resulting formula is, as a rule, quite obscure, non-readable, and because of this practically non-useful. Unlike this, the above block expression in terms of generators is geometrically lucid and well apt for solving procedure of the Schrödinger equation. In various problems it is sufficient to operate algebraically with quantum Poisson brackets. To complete the above system of brackets let us quote expressions involving generators and position-type variables. The latter ones are operators which multiply wave functions by other functions on the configuration space, i.e., $(\mathbf{F}\Psi)(q) := F(q)\Psi(q)$. If there is no danger of misunderstanding, we will not distinguish graphically between \mathbf{F} and F . Just as on the classical level we have

$$_Q\{\Sigma_\mu, \mathbf{F}\} = -\mathbf{L}_\mu F, \quad _Q\{\hat{\Sigma}_\mu, \mathbf{F}\} = -\mathbf{R}_\mu F.$$

Obviously, two position-type operators mutually commute.

Remark: Obviously, only for generators and position quantities the quantum and classical Poisson rules are identical. For other quantities it is no longer the case, moreover, there are problems with the very definition of quantum counterparts of other classical quantities. The very existence of the above distinguished family of physical quantities is due to the group-theoretical background of degrees of freedom.

9 Corresponding Haar measures

Let us now return to the main subject of our analysis, i.e., to the quantization of affine systems. For technical purposes we again fix some Cartesian coordinates x^i , a^K in M , N and identify analytically the configuration space $Q = \text{LI}(U, V) \times M$ with the affine group $\text{GAf}(n, \mathbb{R}) \simeq \text{GL}(n, \mathbb{R}) \times_s \mathbb{R}^n$. Similarly, the internal configuration space $Q_{\text{int}} = \text{LI}(U, V)$ is identified with $\text{GL}(n, \mathbb{R})$. The corresponding Haar measures will be denoted respectively by α , λ , i.e., $d\alpha(\varphi, x) = (\det \varphi)^{-n-1} dx^1 \cdots dx^n d\varphi^1_1 \cdots d\varphi^n_n = (\det \varphi)^{-1} d\lambda(\varphi) dx^1 \cdots dx^n$,

$d\lambda(\varphi) = (\det \varphi)^{-n} d\varphi^1_1 \cdots d\varphi^n_n$. In terms of the binary decomposition we have the following expression:

$$d\lambda(\varphi) = d\lambda(l; q; r) = \prod_{i \neq j} |\text{sh}(q^i - q^j)| d\mu(l) d\mu(r) dq^1 \cdots dq^n,$$

where μ denotes the Haar measure on $\text{SO}(n, \mathbb{R})$. Due to the compactness of $\text{SO}(n, \mathbb{R})$ we can, but of course need not, normalize μ to unity, $\mu(\text{SO}(n, \mathbb{R})) = 1$.

The Haar measure on $\text{SL}(n, \mathbb{R})$ used in quantum mechanics of incompressible objects may be symbolically written with the use of Dirac distribution as follows:

$$d\lambda_{\text{SL}}(\varphi) = \prod_{i \neq j} |\text{sh}(q^i - q^j)| d\mu(l) d\mu(r) \delta(q^1 + \cdots + q^n) dq^1 \cdots dq^n.$$

10 Kinetic energy operators for affine models

Affine spin and its co-moving representation are, respectively, given by the following formally self-adjoint operators:

$$\Sigma^a_b := \frac{\hbar}{i} \mathbf{L}^a_b = \frac{\hbar}{i} \varphi^a_K \frac{\partial}{\partial \varphi^b_K}, \quad \hat{\Sigma}^A_B := \frac{\hbar}{i} \mathbf{R}^A_B = \frac{\hbar}{i} \varphi^m_B \frac{\partial}{\partial \varphi^m_A}.$$

The usual spin and vorticity operators are respectively given by

$$\mathbf{S}^a_b := \Sigma^a_b - g^{ac} g_{bd} \Sigma^d_c, \quad \mathbf{V}^A_B := \hat{\Sigma}^A_B - \eta^{AC} \eta_{BD} \hat{\Sigma}^D_C. \quad (4)$$

Kinetic energy operators corresponding to the formerly described classical models of internal kinetic energies are simply obtained by replacing the classical quantities Σ^a_b , $\hat{\Sigma}^A_B$ by the above operators Σ^a_b , $\hat{\Sigma}^A_B$ without any attention to be paid to the ordering problem (just because of the group-theoretic interpretation of these quantities).

Thus, for the affine-affine model (affine both in space and in the material) we have

$$\begin{aligned} \mathbf{T}^{\text{aff-aff}}_{\text{int}} &= \frac{1}{2A} \Sigma^i_j \Sigma^j_i - \frac{B}{2A(A+nB)} \Sigma^i_i \Sigma^j_j \\ &= \frac{1}{2A} \hat{\Sigma}^A_B \hat{\Sigma}^B_A - \frac{B}{2A(A+nB)} \hat{\Sigma}^A_A \hat{\Sigma}^B_B. \end{aligned}$$

Similarly, for models with the mixed metrical-affine and affine-metrical invariance we have, respectively,

$$\begin{aligned} \mathbf{T}^{\text{met-aff}}_{\text{int}} &= \frac{1}{2\tilde{I}} g_{ik} g^{jl} \Sigma^i_j \Sigma^k_l + \frac{1}{2\tilde{A}} \Sigma^i_j \Sigma^j_i + \frac{1}{2\tilde{B}} \Sigma^i_i \Sigma^j_j, \\ \mathbf{T}^{\text{aff-met}}_{\text{int}} &= \frac{1}{2\tilde{I}} \eta_{AB} \eta^{CD} \hat{\Sigma}^A_C \hat{\Sigma}^B_D + \frac{1}{2\tilde{A}} \hat{\Sigma}^A_B \hat{\Sigma}^B_A + \frac{1}{2\tilde{B}} \hat{\Sigma}^A_A \hat{\Sigma}^B_B, \end{aligned}$$

where $\tilde{I} = (I^2 - A^2)/I$, $\tilde{A} = (A^2 - I^2)/A$, $\tilde{B} = -(I + A)(I + A + nB)/B$

Similarly, the corresponding expressions for \mathcal{T}_{tr} have the following forms:

$$\begin{aligned}\mathbf{T}_{\text{tr}}^{\text{met-aff}} &= \frac{m}{2} g^{ij} \mathbf{P}_i \mathbf{P}_j = \frac{m}{2} \tilde{G}^{AB} \hat{\mathbf{P}}_A \hat{\mathbf{P}}_B, \\ \mathbf{T}_{\text{tr}}^{\text{aff-met}} &= \frac{m}{2} \tilde{C}^{ij} \mathbf{P}_i \mathbf{P}_j = \frac{m}{2} \eta^{AB} \hat{\mathbf{P}}_A \hat{\mathbf{P}}_B,\end{aligned}$$

where \mathbf{P}_i , $\hat{\mathbf{P}}_A$ are linear momentum operators respectively in laboratory and co-moving representations,

$$\mathbf{P}_a = \frac{\hbar}{i} \frac{\partial}{\partial x^a}, \quad \hat{\mathbf{P}}_K = \varphi^a_K \mathbf{P}_a = \frac{\hbar}{i} \varphi^a_K \frac{\partial}{\partial x^a}.$$

Just as previously, \tilde{C} , \tilde{G} are contravariant reciprocals of deformation tensors: $\tilde{C}^{ik} C_{kj} = \delta^i_j$, $\tilde{G}^{AC} G_{CB} = \delta^A_B$. As mentioned, there are no affine-affine models of \mathbf{T}_{tr} , and therefore, no affine-affine models of \mathbf{T} . The corresponding "metric tensors" on $\text{Gaf}(n, \mathbb{R})$ would have to be singular.

Another important physical quantity is the canonical momentum conjugate to the dilatational coordinate q . On the quantum level it is represented by the formally self-adjoint operator

$$\mathbf{p} = \frac{\hbar}{i} \frac{\partial}{\partial q}.$$

It is also convenient to use the deviatoric (shear) parts of the affine spin,

$$\mathbf{s}^a_b := \Sigma^a_b - \frac{\mathbf{p}}{n} \delta^a_b, \quad \hat{\mathbf{s}}^A_B := \hat{\Sigma}^A_B - \frac{\mathbf{p}}{n} \delta^A_B;$$

obviously, $\mathbf{p} = \Sigma^a_a = \hat{\Sigma}^A_A$.

Due to the group-theoretical structure of the above objects as generators, the classical splitting of \mathbf{T} into incompressible (shear-rotational) and dilatational parts remains literally valid, namely, we have the following expressions:

$$\begin{aligned}\mathbf{T}_{\text{int}}^{\text{aff-aff}} &= \frac{1}{2A} \mathbf{C}_{\text{SL}(n)}(2) + \frac{1}{2n(A+nB)} \mathbf{p}^2, \\ \mathbf{T}_{\text{int}}^{\text{met-aff}} &= \frac{1}{2(I+A)} \mathbf{C}_{\text{SL}(n)}(2) + \frac{1}{2n(I+A+nB)} \mathbf{p}^2 + \frac{I}{2(I^2-A^2)} \|\mathbf{S}\|^2, \\ \mathbf{T}_{\text{int}}^{\text{aff-met}} &= \frac{1}{2(I+A)} \mathbf{C}_{\text{SL}(n)}(2) + \frac{1}{2n(I+A+nB)} \mathbf{p}^2 + \frac{I}{2(I^2-A^2)} \|\mathbf{V}\|^2,\end{aligned}$$

where, obviously, $\mathbf{C}_{\text{SL}(n)}(k) := \mathbf{s}^a_b \mathbf{s}^b_c \cdots \mathbf{s}^r_s \mathbf{s}^s_a = \hat{\mathbf{s}}^A_B \hat{\mathbf{s}}^B_C \cdots \hat{\mathbf{s}}^R_S \hat{\mathbf{s}}^S_A$, k terms in these expressions, and $\|\mathbf{S}\|^2 = -(1/2) \mathbf{S}^a_b \mathbf{S}^b_a$, $\|\mathbf{V}\|^2 = -(1/2) \mathbf{V}^A_B \mathbf{V}^B_A$.

As mentioned, the $\text{SL}(n, \mathbb{R})$ -part of \mathbf{T} has both discrete and continuous spectrum and predicts the bounded oscillatory solutions even if no extra potential on $\text{SL}(n, \mathbb{R})$ is used (classically this is the geodetic model with an open subset of bounded trajectories in the complete solution). In particular, there is an open range of inertial parameters $(A, B, C) \in \mathbb{R}^3$ for which the spectrum is positive or at least bounded from below.

One can hope that on the basis of commutation relations for the Lie algebra $\text{SL}(n, \mathbb{R})'$ some information concerning spectra and wave functions may be perhaps obtained without the explicit solving of differential equations.

There are $\text{GL}(n, \mathbb{R})$ -problems where the separation of the isochoric $\text{SL}(n, \mathbb{R})$ -terms is not necessary, sometimes it is even undesirable. Then it is more convenient to use the quantized version of ([20]4.45)¹, ([20]4.46), ([20]4.47), i.e.,

$$\begin{aligned}\mathbf{T}_{\text{int}}^{\text{aff-aff}} &= \frac{1}{2A}\mathbf{C}(2) - \frac{B}{2A(A+nB)}\mathbf{p}^2, \\ \mathbf{T}_{\text{int}}^{\text{met-aff}} &= \frac{1}{2\alpha}\mathbf{C}(2) + \frac{1}{2\beta}\mathbf{p}^2 + \frac{1}{2\mu}\|\mathbf{S}\|^2, \\ \mathbf{T}_{\text{int}}^{\text{aff-met}} &= \frac{1}{2\alpha}\mathbf{C}(2) + \frac{1}{2\beta}\mathbf{p}^2 + \frac{1}{2\mu}\|\mathbf{V}\|^2,\end{aligned}$$

where $\alpha := I + A$, $\beta := -(I + A)(I + A + nB)/B$, $\mu := (I^2 - A^2)/I$, and $\mathbf{C}(k)$ are operators of the full $\text{GL}(n, \mathbb{R})$ -Casimirs, i.e., we have

$$\mathbf{C}(k) := \Sigma^a{}_b \Sigma^b{}_c \cdots \Sigma^r{}_s \Sigma^s{}_a = \hat{\Sigma}^A{}_B \hat{\Sigma}^B{}_C \cdots \hat{\Sigma}^R{}_S \hat{\Sigma}^S{}_A;$$

the above contracted products contain k terms. In particular,

$$\mathbf{C}(2) := \Sigma^a{}_b \Sigma^b{}_a = \hat{\Sigma}^A{}_B \hat{\Sigma}^B{}_A, \quad \mathbf{C}(1) := \Sigma^a{}_a = \hat{\Sigma}^A{}_A.$$

In particular, if the inertial constant B vanishes, then the model $\mathbf{T}_{\text{int}}^{\text{aff-aff}}$ may be interpreted in terms of one-dimensional multi-body problems in the sense of Calogero, Moser, Sutherland [15, 21], etc., quite independently of our primary motivation, i.e., n -dimensional affine systems.

As mentioned, on $\text{GL}(n, \mathbb{R})$, i.e., for compressible objects with dilatations, some dilatation-stabilizing potential $V(q)$ must be introduced if the system has to possess bound states. For more general doubly isotropic potentials $V(q^1, \dots, q^n)$ depending only on deformation invariants, there is no possibility of avoiding differential equations (with the help of ladder procedures). Nevertheless, the problem is then still remarkably simplified in comparison with the general case, because the quantum dynamics of deformation invariants is autonomous (in this respect the quantum problem is in a sense simpler than the classical one). The procedure is based then on the two-polar decomposition, which by the way is also very convenient on the level of purely geodetic models. In certain problems, e.g., spatially isotropic but materially anisotropic ones, the polar decomposition is also convenient.

11 Two-polar decomposition in quantum case

Let us go back to classical expressions for $\hat{\rho}, \hat{\tau} \in \text{SO}(n, \mathbb{R})'$, $\rho \in \text{SO}(V, g)'$, $\tau \in \text{SO}(U, \eta)'$, $M := -\hat{\rho} - \hat{\tau}$, $N := \hat{\rho} - \hat{\tau}$. On the quantum level the classical

¹this kind of references means that, e.g., in Part I [20] the expression could be found in section 4 with label 45

quantities $\rho = S$, $\tau = -V$ become the operators of spin and minus vorticity (4) \mathbf{S} , $-\mathbf{V}$, i.e., Hermitian generators of the unitary groups of spatial and material rotations $\varphi \mapsto A\varphi$, $\varphi \mapsto \varphi B^{-1}$, where $A \in \text{SO}(V, g)$, $B \in \text{SO}(U, \eta)$, acting argument-wise on wave functions. Classical quantities $\hat{\rho}$, $\hat{\tau}$ were co-moving representants of tensors $\rho = S$, $\tau = -V$, i.e., their projections onto principal axes of the Cauchy and Green deformation tensors. Their quantum counterparts, i.e., operators $\hat{\mathbf{r}}$, $\hat{\mathbf{t}}$ are also co-moving representants of $\mathbf{r} = \mathbf{S}$, $\mathbf{t} = -\mathbf{V}$, i.e.,

$$\hat{\mathbf{r}}^a_b = L^a_i L^j_b \mathbf{S}^i_j, \quad \hat{\mathbf{t}}^a_b = -R^A_b R^a_B \mathbf{V}^B_A. \quad (5)$$

They are Hermitian generators of the argument-wise right-hand side action ([20]6.63) of $\text{SO}(n, \mathbb{R})$ on the wave functions. Just as in classical theory, it is convenient to introduce operators

$$\mathbf{M}^a_b := -\hat{\mathbf{r}}^a_b - \hat{\mathbf{t}}^a_b, \quad \mathbf{N}^a_b := \hat{\mathbf{r}}^a_b - \hat{\mathbf{t}}^a_b. \quad (6)$$

Commutation relations for operators \mathbf{S} , \mathbf{V} , $\hat{\mathbf{r}}$, $\hat{\mathbf{t}}$, \mathbf{M} , \mathbf{N} are directly isomorphic with those for the generators of $\text{SO}(n, \mathbb{R})$ and are expressed in a straightforward way in terms of $\text{SO}(n, \mathbb{R})$ -structure constants.

Now we are ready to write down explicitly our kinetic energy and Hamiltonian operators in terms of the two-polar splitting. We begin with the traditional integer spin models, and later on we show how half-integer angular momentum of extended bodies may appear in a natural way.

Quantum operators \mathbf{S}^i_j , $-\mathbf{V}^A_B$ have the following form:

$$\mathbf{S}^i_j = \frac{\hbar}{i} \mathbf{\Lambda}^i_j(L), \quad -\mathbf{V}^A_B = \frac{\hbar}{i} \mathbf{\Lambda}^A_B(R), \quad (7)$$

where, according to the formulas (1), (2), (3), $\mathbf{\Lambda}^i_j(L)$ and $\mathbf{\Lambda}^A_B(R)$ are real first-order differential operators generating left regular translations on $\text{SO}(n, \mathbb{R})$, or, more precisely, on the isometric factors $L : \mathbb{R}^n \rightarrow V$, $R : \mathbb{R}^n \rightarrow U$ of the two-polar splitting, i.e.,

$$\begin{aligned} F(W(\omega)L) &= \left(\exp \left(\frac{1}{2} \omega^j_i \mathbf{\Lambda}^i_j \right) F \right) (L), \\ F(W(\omega)R) &= \left(\exp \left(\frac{1}{2} \omega^B_A \mathbf{\Lambda}^A_B \right) F \right) (R). \end{aligned} \quad (8)$$

In the formulas above, F are functions on the manifolds of isometries from (\mathbb{R}^n, δ) to (V, g) and from (\mathbb{R}^n, δ) to (U, η) . Analytically, in Cartesian coordinates they are simply functions on $\text{SO}(n, \mathbb{R})$. Matrices $[\omega^a_b]$, $[\omega^A_B]$ are respectively g - and η -antisymmetric: $\omega^a_b = -g^{ac} g_{bd} \omega^d_c$, $\omega^A_B = -\eta^{AC} \eta_{BD} \omega^D_C$. Their independent components are canonical coordinates of the first kind on $\text{SO}(V, g)$, $\text{SO}(U, \eta)$ (roughly, on $\text{SO}(n, \mathbb{R})$),

$$W(\omega) = \exp \left(\frac{1}{2} \omega^b_a E^a_b \right), \quad W(\omega) = \exp \left(\frac{1}{2} \omega^B_A E^A_B \right), \quad (9)$$

where $E^a{}_b \in \text{SO}(V, g)'$, $E^A{}_B \in \text{SO}(U, \eta)'$ are basic elements corresponding to some (arbitrary) choice of bases in V , U , i.e., $(E^a{}_b)^i{}_j = \delta^a_j \delta^i_b - g^{ai} g_{bj}$, $(E^A{}_B)^C{}_D = \delta^A_D \delta^C_B - \eta^{AC} \eta_{BD}$.

One could reproach against our permanent changing between the simplified analytical description based on \mathbb{R}^n , $\text{GL}^+(n, \mathbb{R})$, $\text{SO}(n, \mathbb{R})$ and the careful geometric distinguishing between the material and physical spaces U , V and the manifolds $\text{LI}(U, V)$, $\text{O}^+(\mathbb{R}^n, \delta; V, g)$, $\text{O}^+(\mathbb{R}^n, \delta; U, \eta)$; the latter two denoting the manifolds of orientation-preserving isometries between indicated Euclidean spaces (equivalently, manifolds of positively oriented orthonormal frames $\text{F}^+(V, g)$, $\text{F}^+(U, \eta)$). However, this "monkey" way of changing branches has some advantages, provided that done carefully. There are relationships easily representable for computational purposes in matrix terms, however, in certain fundamental formulas this may be misleading and risky.

And now, at some final stage of our discussion there appear some expressions where the calculus on \mathbb{R}^n as such (not on \mathbb{R}^n base-identified with U , V) becomes not only temporarily admissible but just mathematically proper one. Namely, it is just the matrix group $\text{SO}(n, \mathbb{R})$ that acts on the right on the objects $L \in \text{O}^+(\mathbb{R}^n, \delta; V, g)$ and $R \in \text{O}^+(\mathbb{R}^n, \delta; U, \eta)$. As said above, on the classical level the corresponding Hamiltonian generators, i.e., momentum mappings, are given by $[\hat{\rho}^a{}_b]$, $[\hat{\tau}^a{}_b]$. In quantized theory the same role is played by the formally self-adjoint differential operators $\hat{\mathbf{r}}^a{}_b$, $\hat{\mathbf{t}}^a{}_b$,

$$\begin{aligned} F(LW(\omega)) &= \left(\exp \left(\frac{1}{2} \omega^b{}_a \Upsilon^a{}_b \right) F \right) (L) = \left(\exp \left(\frac{i}{2\hbar} \omega^b{}_a \hat{\mathbf{r}}^a{}_b \right) F \right) (L) \\ F(RW(\omega)) &= \left(\exp \left(\frac{1}{2} \omega^b{}_a \Upsilon^a{}_b \right) F \right) (R) = \left(\exp \left(\frac{i}{2\hbar} \omega^b{}_a \hat{\mathbf{t}}^a{}_b \right) F \right) (R). \end{aligned} \quad (10)$$

Here the skew-symmetry of $[\omega^a{}_b]$ is meant in the literal Kronecker-delta sense; nothing like g and η is implicitly assumed: $\omega^a{}_b = -\omega_b{}^a = -\delta^{ac} \delta_{bd} \omega^d{}_c$. Just \mathbb{R}^n as such with its numerical metric is used here. In the physical three-dimensional case one uses the duality between skew-symmetric tensors and axial vectors, thus, on the quantum operator level we use the quantities $\hat{\mathbf{r}}_a$, $\hat{\mathbf{t}}_a$, $\Upsilon(L)_a$, $\Upsilon(R)_a$, where

$$\begin{aligned} \hat{\mathbf{r}}^a{}_b &= \epsilon^a{}_b{}^c \hat{\mathbf{r}}_c, & \hat{\mathbf{t}}^a{}_b &= \epsilon^a{}_b{}^c \hat{\mathbf{t}}_c, & \Upsilon^a{}_b &= \epsilon^a{}_b{}^c \Upsilon_c, \\ \hat{\mathbf{r}}_a &= \frac{1}{2} \epsilon_{ab}{}^c \hat{\mathbf{r}}^b{}_c, & \hat{\mathbf{t}}_a &= \frac{1}{2} \epsilon_{ab}{}^c \hat{\mathbf{t}}^b{}_c, & \Upsilon_a &= \frac{1}{2} \epsilon_{ab}{}^c \Upsilon^b{}_c. \end{aligned}$$

Obviously, the expressions $\Upsilon^a{}_b$, Υ_a are meant in two versions, as acting on the L, R -variables, thus, puristically we should have used the symbols $\Upsilon^a{}_b(L)$, $\Upsilon_a(L)$, $\Upsilon^a{}_b(R)$, $\Upsilon_a(R)$, however, when non-necessary, we prefer to avoid the crowd of symbols. Commutation relations are in both cases: $[\Upsilon_a, \Upsilon_b] = \epsilon_{ab}{}^c \Upsilon_c$, i.e., in terms of quantum Poisson brackets:

$$\frac{1}{i\hbar} [\hat{\mathbf{r}}_a, \hat{\mathbf{r}}_b] = -\epsilon_{ab}{}^c \hat{\mathbf{r}}_c, \quad \frac{1}{i\hbar} [\hat{\mathbf{t}}_a, \hat{\mathbf{t}}_b] = -\epsilon_{ab}{}^c \hat{\mathbf{t}}_c.$$

It is clear that $[\hat{\mathbf{r}}_a, \hat{\mathbf{t}}_b] = 0$, $[\mathbf{\Upsilon}_a(L), \mathbf{\Upsilon}_b(R)] = 0$. Obviously, the raising and lowering of indices is meant here in the trivial Kronecker-delta sense, so it is written only for cosmetic reasons, e.g., $\epsilon^a_b{}^c = \delta^{ak}\delta^{cl}\epsilon_{kbl}$, etc. What concerns the V - and U -space objects like $\mathbf{S}^i_j = \mathbf{r}^i_j$, $\mathbf{V}^A_B = -\mathbf{t}^A_B$, analogous expressions are true when one uses orthonormal coordinates, i.e., when $g_{ij} =_* \delta_{ij}$, $\eta_{AB} =_* \delta_{AB}$. When more general rectilinear coordinates are used, the formulas become more complicated because various expressions involving $\det[g_{ij}]$, $\det[\eta_{AB}]$ appear; there is, however, no practical need to use this representation.

In orthonormal coordinates in V and U spaces we have again the following expressions in terms of axial vectors: $\mathbf{r}^i_j = \mathbf{S}^i_j = \epsilon^i_j{}^k \mathbf{r}_k = \epsilon^i_j{}^k \mathbf{S}_k$, $\mathbf{t}^A_B = -\mathbf{V}^A_B = \epsilon^A_B{}^C \mathbf{t}_C = -\epsilon^A_B{}^C \mathbf{V}_C$. These quantities are expressed through differential operators $\mathbf{\Lambda}^i_j(L)$, $\mathbf{\Lambda}^A_B(R)$, cf. (7), for which the same dual representation will be used, i.e.,

$$\mathbf{\Lambda}^i_j(L) = \epsilon^i_j{}^k \mathbf{\Lambda}_k(L), \quad \mathbf{\Lambda}_k(L) = \frac{1}{2} \epsilon_{ij}{}^k \mathbf{\Lambda}^j_k(L),$$

$$\mathbf{\Lambda}^A_B(R) = \epsilon^A_B{}^C \mathbf{\Lambda}_C(R), \quad \mathbf{\Lambda}_A(R) = \frac{1}{2} \epsilon_{AB}{}^C \mathbf{\Lambda}^B_C(R).$$

When using the convention of "small" and "capital" indices, one can omit the L - and R -labels at $\mathbf{\Lambda}$ -symbols. Obviously, we have $\mathbf{S}_i = \mathbf{r}_i = (\hbar/i)\mathbf{\Lambda}_i$, $\mathbf{V}_A = -\mathbf{t}_A = -(\hbar/i)\mathbf{\Lambda}_A$. One should be careful with some subtle sign problems in commutation relations,

$$[\mathbf{\Lambda}_i, \mathbf{\Lambda}_j] = -\epsilon_{ij}{}^k \mathbf{\Lambda}_k, \quad [\mathbf{\Lambda}_A, \mathbf{\Lambda}_B] = -\epsilon_{AB}{}^C \mathbf{\Lambda}_C, \quad [\mathbf{\Lambda}_i, \mathbf{\Lambda}_A] = 0,$$

therefore,

$$\frac{1}{i\hbar}[\mathbf{S}_i, \mathbf{S}_j] = \epsilon_{ij}{}^k \mathbf{S}_k, \quad \frac{1}{i\hbar}[\mathbf{V}_A, \mathbf{V}_B] = -\epsilon_{AB}{}^C \mathbf{V}_C, \quad [\mathbf{S}_i, \mathbf{V}_A] = 0.$$

Let us also notice that

$$\begin{aligned} [\mathbf{\Lambda}_i, \mathbf{\Upsilon}_a(L)] &= 0, & [\mathbf{\Lambda}_i, \mathbf{\Lambda}_A] &= 0, & [\mathbf{\Lambda}_i, \mathbf{\Upsilon}_a(R)] &= 0, \\ [\mathbf{\Lambda}_A, \mathbf{\Upsilon}_a(L)] &= 0, & [\mathbf{\Upsilon}_a(L), \mathbf{\Upsilon}_a(R)] &= 0, & [\mathbf{\Lambda}_A, \mathbf{\Upsilon}_a(R)] &= 0. \end{aligned}$$

12 Rotation-vector-space language

Obviously, "coordinates" ω^a_b on $\text{SO}(n, \mathbb{R})$ are redundant, unless we restrict ourselves to $\omega_{ab} = \delta_{ac}\omega^c_b$, $a < b$ (or conversely). If $n = 3$, one uses so-called "rotation vector" k^a , where $\omega^a_b = -\epsilon^a_{bc}k^c$, $k^a = -(1/2)\epsilon^a_b{}^c\omega^b_c$. It is convenient to use the "magnitude" $k = \sqrt{(k^1)^2 + (k^2)^2 + (k^3)^2}$. In this parameterization, $\text{SO}(3, \mathbb{R})$ is covered by the ball $k \leq \pi$ with the proviso that antipodal points on the sphere $k = \pi$ describe the same half-rotation, i.e., rotation by π about a given axis. For $k < \pi$ the representation is unique. The magnitude k equals the angle of rotation, whereas the versor $\bar{n} := \bar{k}/k$ represents the oriented rotation

axis in the right screw sense (for $k = \pi$ it does not matter right or left ones; they coincide). In certain expressions it is convenient to use the spherical coordinates k, ϑ, φ in the \bar{k} -space, thus, $k^1 = k \sin \vartheta \cos \varphi$, $k^2 = k \sin \vartheta \sin \varphi$, $k^3 = k \cos \vartheta$. For the completeness, let us quote some important three-dimensional formulas.

The "basic" matrices $E^a_b \in \text{SO}(3, \mathbb{R})'$ are represented dually by the actually basic system of E_a , where $E^a_b = \epsilon^a_b{}^c E_c$, $E_a = (1/2)\epsilon_{ab}{}^c E^b_c$, $(E_a)^b_c = -\epsilon_a{}^b{}_c$. The structure constants are then given simply by "epsilons": $[E_a, E_b] = \epsilon_{ab}{}^c E_c$. For any rotation vector $\bar{k} \in \mathbb{R}^3$ the corresponding matrices $W(\bar{k}) \in \text{SO}(3, \mathbb{R})$ act on vectors $\bar{u} \in \mathbb{R}^3$ as follows:

$$W(\bar{k}) \cdot \bar{u} = \cos k \bar{u} + \frac{(1 - \cos k)}{k^2} (\bar{k} \cdot \bar{u}) \bar{k} + \frac{\sin k}{k} \bar{k} \times \bar{u};$$

obviously, the scalar and vector product are meant in the standard \mathbb{R}^3 -sense. The components of \bar{k} are canonical coordinates of the first kind on $\text{SO}(3, \mathbb{R})$,

$$W(\bar{k}) = \exp(k^a E_a) = \sum_{m=0}^{\infty} \frac{1}{m!} (k^a E_a)^m.$$

One can show that

$$W(\bar{k}) \cdot \bar{u} = \bar{u} + \bar{k} \times \bar{u} + \frac{1}{2} \bar{k} \times (\bar{k} \times \bar{u}) + \dots + \frac{1}{n!} \bar{k} \times (\bar{k} \times (\bar{k} \times \dots (\bar{k} \times \bar{u}) \dots)) + \dots$$

This infinite series is an alternative representation of the exponential formula. The term with multiplier $1/n!$ contains the n -fold vector multiplication of \bar{u} by \bar{k} . Explicitly the matrix of $W(\bar{k})$ is given by

$$W(\bar{k})^a{}_b = \cos k \delta^a_b + (1 - \cos k) \frac{k^a k_b}{k^2} + \sin k \epsilon^a{}_{bc} \frac{k^c}{k};$$

obviously, the raising and lowering of indices is meant here in the trivial (purely cosmetic) delta-sense.

One can show that the generators of right regular translations on $\text{SO}(3, \mathbb{R})$ are given by the following expression:

$$\Upsilon_a = \frac{k}{2} \text{ctg} \frac{k}{2} \frac{\partial}{\partial k^a} + \left(1 - \frac{k}{2} \text{ctg} \frac{k}{2}\right) \frac{k_a k^b}{k^2} \frac{\partial}{\partial k^b} - \frac{1}{2} \epsilon_{ab}{}^c k^b \frac{\partial}{\partial k^c}.$$

This is a common formula for $\Upsilon_a(L)$, $\Upsilon_a(R)$, and now for simplicity we again use the analytical matrix representation, when U and V are identified with \mathbb{R}^3 and the L, R -terms of the two-polar decomposition are identified with elements of $\text{SO}(3, \mathbb{R})$. To specify this formula to $\Upsilon_a(L)$, $\Upsilon_a(R)$ one must replace the general symbol of the rotation vector \bar{k} on $\text{SO}(3, \mathbb{R})$ by the rotation vectors \bar{l} , \bar{r} parameterizing the L, R -terms: $L(\bar{l}) = \exp(l^a E_a)$, $R(\bar{r}) = \exp(r^a E_a)$. Generators of the left regular translations on $\text{SO}(3, \mathbb{R})$ are as follows:

$$\Lambda_a = \frac{k}{2} \text{ctg} \frac{k}{2} \frac{\partial}{\partial k^a} + \left(1 - \frac{k}{2} \text{ctg} \frac{k}{2}\right) \frac{k_a k^b}{k^2} \frac{\partial}{\partial k^b} + \frac{1}{2} \epsilon_{ab}{}^c k^b \frac{\partial}{\partial k^c}.$$

And this again specifies to $\Lambda_a(L)$, $\Lambda_a(R)$ when instead of \bar{k} we substitute respectively \bar{l} , \bar{r} , i.e., rotation vectors parameterizing the manifolds of L , R -factors in the two-polar decomposition.

Let us observe that $\Lambda_a - \Upsilon_a = \mathbf{D}_a = \epsilon_{ab}{}^c k^b (\partial/\partial k^c)$, and these differential operators generate the group of inner automorphisms of $\text{SO}(3, \mathbb{R})$: $W(\bar{k}) \mapsto UW(\bar{k})U^{-1} = W(U\bar{k})$, where U runs over $\text{SO}(3, \mathbb{R})$. Roughly speaking, these transformations result in rotations of the rotation vectors. And, just as previously, substituting here \bar{l} and \bar{r} in place of \bar{k} we obtain the corresponding transformations of the manifolds of $L(\bar{l})$ - and $R(\bar{r})$ -terms of the two-polar decompositions. One can show that the generators of the left and right regular translations on $\text{SO}(3, \mathbb{R})$ may be expressed in terms of operators $\partial/\partial k$ and \mathbf{D}_a acting, respectively, along the radius and tangentially to spheres in the representative spaces \mathbb{R}^3 of the rotation vector \bar{k} , i.e.,

$$\Lambda_a = \frac{k_a}{k} \frac{\partial}{\partial k} - \frac{1}{2} \text{ctg} \frac{k}{2} \epsilon_{ab}{}^c k^b \mathbf{D}_c + \frac{1}{2} \mathbf{D}_a, \quad \Upsilon_a = \frac{k_a}{k} \frac{\partial}{\partial k} - \frac{1}{2} \text{ctg} \frac{k}{2} \epsilon_{ab}{}^c k^b \mathbf{D}_c - \frac{1}{2} \mathbf{D}_a.$$

Obviously, $[\mathbf{D}_a, \mathbf{D}_b] = -\epsilon_{ab}{}^c \mathbf{D}_c$.

In many formulas we need orthogonal invariants like $\|\mathbf{S}\|^2$, $\|\mathbf{V}\|^2$. They are based on the Casimir invariants $C_{\text{SO}(n, \mathbb{R})}(2)$ built of generators Λ_a , Υ_a of the left and right regular translations on $\text{SO}(n, \mathbb{R})$. If $n = 3$, these Casimirs have the following form:

$$\Lambda^2 = \Upsilon^2 = \Lambda_1^2 + \Lambda_2^2 + \Lambda_3^2 = \Upsilon_1^2 + \Upsilon_2^2 + \Upsilon_3^2, \quad (11)$$

and one can show that analytically

$$\mathbf{C}_{\text{SO}(3, \mathbb{R})}(2) = \Lambda^2 = \Upsilon^2 = \left(\frac{\partial^2}{\partial k^2} + \text{ctg} \frac{k}{2} \frac{\partial}{\partial k} \right) + \frac{1}{4 \sin^2 \frac{k}{2}} \mathbf{D}^2,$$

where $\mathbf{D}^2 = \mathbf{D}_1^2 + \mathbf{D}_2^2 + \mathbf{D}_3^2$. Obviously, $\|\mathbf{S}\|^2 = -\hbar^2 \mathbf{C}_{\text{SO}(3, \mathbb{R})}(L(\bar{l}))$ and $\|\mathbf{V}\|^2 = -\hbar^2 \mathbf{C}_{\text{SO}(3, \mathbb{R})}(R(\bar{r}))$, where the last two terms multiplied by $-\hbar^2$ are obtained from the previous $\mathbf{C}_{\text{SO}(3, \mathbb{R})}$ by substituting the \bar{l} - and \bar{r} -variables in place of \bar{k} .

Remark: Obviously, the equality (11) of Λ^2 and Υ^2 holds only when Λ_a and Υ_a involve the same kind of independent variables, e.g., \bar{k} on the abstract $\text{SO}(3, \mathbb{R})$ as generators of the left or right regular translations, \bar{l} when both operating on the left two-polar factor $L(\bar{l})$, or \bar{r} when both acting on the right two-polar factor. But of course $\|\mathbf{S}\|^2$ and $\|\mathbf{V}\|^2$ are different for any dimension n , although, of course, $\|\mathbf{S}\|^2 = \|\hat{\mathbf{r}}\|^2$ and $\|\mathbf{V}\|^2 = \|\hat{\mathbf{t}}\|^2$ always hold just on the basis of equations (5).

13 Expansion of wave functions

When we use the two-polar decomposition $\varphi = LDR^{-1}$, then, according to the Peter-Weyl theorem, the wave functions on $\text{GL}^+(n, \mathbb{R})$ may be expanded in L, R -variables with respect to matrix elements of irreducible representations

of the compact group $\text{SO}(n, \mathbb{R})$. Obviously, the expansion coefficients depend on deformation invariants, i.e., on the diagonal factor D (equivalently, on the variables Q^a or $q^a = \ln Q^a$). In general, we have that

$$\Psi(\varphi) = \Psi(L, D, R) = \sum_{\alpha, \beta \in \Omega} \sum_{m, n=1}^{N(\alpha)} \sum_{k, l=1}^{N(\beta)} \mathcal{D}_{mn}^\alpha(L) f_{ml}^{\alpha\beta}(D) \mathcal{D}_{kl}^\beta(R^{-1}), \quad (12)$$

where the meaning of symbols is as follows:

- Ω is the set of equivalence classes of unitary irreducible representations of $\text{SO}(n, \mathbb{R})$.
- $N(\alpha)$ is the dimension of the α -th representation class. It is finite because $\text{SO}(n, \mathbb{R})$ is compact.
- \mathcal{D}^α is the α -th representation matrix. For many classical groups \mathcal{D}^α are explicitly known (at least in terms of some well-investigated special functions).

Analytically $\mathcal{D}^\alpha(L)$, $\mathcal{D}^\beta(R^{-1})$ are matrices depending on the group coordinates ω_L^{ab} , ω_R^{ab} of L , R , e.g., rotation vectors \vec{l} , \vec{r} if $n = 3$. The argument D of f is the system of q -variables q^1, \dots, q^n . According to the mentioned multi-valuedness of the two-polar decomposition, the reduced amplitudes $f^{\alpha\beta}(q^1, \dots, q^n)$ must obey some conditions, because Ψ must not distinguish triplets (L, D, R) corresponding to the same configuration $\varphi = LDR^{-1}$.

Therefore, on the submanifold $M^{(n)} \subset \text{SO}(n, \mathbb{R}) \times \mathbb{R}^n \times \text{SO}(n, \mathbb{R})$ with non-degenerate systems of (q^1, \dots, q^n) (no coincidences) we must have that

$$f_{ml}^{\alpha\beta}(q^{\pi_W(1)}, \dots, q^n) = \sum_{r=1}^{N(\alpha)} \sum_{s=1}^{N(\beta)} \mathcal{D}_{nr}^\alpha(W^{-1}) f_{ml}^{\alpha\beta}(q^1, \dots, q^n) \mathcal{D}_{sk}^\beta(W)$$

for any $W \in K^+$. The same holds on the subsets $M^{(k;p_1, \dots, p_k)} \subset \text{SO}(n, \mathbb{R}) \times \mathbb{R}^n \times \text{SO}(n, \mathbb{R})$ with degenerate systems (q^1, \dots, q^n) (coincidences of some q 's). The difference is that in degenerate cases W runs over the continuous subgroups of $\text{SO}(n, \mathbb{R})$ generated by K^+ and the subgroups $H^{(k;p_1, \dots, p_k)}$ described above. The special case of the total degeneracy is extreme and, because of this, very simple one. Indeed, then in the two-polar decomposition it is only LR^{-1} that is meaningful whereas L , R separately are not well-defined. Therefore, if $D = cI_n$, i.e., $q^1 = \dots = q^n = q$, then the reduced amplitude obeys very severe restrictions, i.e., $f^{\alpha\beta}(cI_n) = 0$ if $\alpha \neq \beta$, and $f_{ml}^{\alpha\alpha}(cI_n) = g_{ml} \delta_{rs}$. The non-uniqueness is extreme here, namely, for any $Z \in \text{SO}(n, \mathbb{R})$ the triplets (L, cI_n, R) , (LZ, cI_n, RZ) represent the same classical configuration, thus, the wave functions do not distinguish them.

It is seen that if q^1, \dots, q^n are interpreted as coordinates of some fictitious material points on the real axis \mathbb{R} , one is dealing with a very peculiar system of identical para-statistical particles.

It is clear that in geodetic models or in models with doubly isotropic potentials (ones depending only on deformation invariants; dilatation-stabilizing potentials $V(q)$ provide the simplest example), m and l in the Peter-Weyl expansion (12) are "good" quantum numbers. In other words, the spin and vorticity operators \mathbf{S}^i_j , \mathbf{V}^A_B do commute with the Hamilton operator \mathbf{H} . The same concerns representation labels $\alpha, \beta \in \Omega$, i.e., finally, the systems of eigenvalues for the Casimir operators of the groups $\text{SO}(V, g)$, $\text{SO}(U, \eta)$ acting argument-wise on wave functions. Let us remind that these Casimirs are given by

$$\mathbf{C}_{\text{SO}(V, g)}(p) \simeq \mathbf{S}^i_k \mathbf{S}^k_m \cdots \mathbf{S}^r_z \mathbf{S}^z_i, \quad \mathbf{C}_{\text{SO}(U, \eta)}(p) \simeq \mathbf{V}^A_K \mathbf{V}^K_M \cdots \mathbf{V}^R_Z \mathbf{V}^Z_A, \quad (13)$$

p operator multipliers in every expression; $p \leq n$ and even.

In such situation it is convenient to keep α, β, m, l fixed and use the following reduced amplitudes (with the same as previously provisos concerning the one-valuedness of Ψ as a function of φ):

$$\Psi(\varphi) = \Psi_{ml}^{\alpha\beta}(L, D, R) = \sum_{n=1}^{N(\alpha)} \sum_{k=1}^{N(\beta)} \mathcal{D}_{mn}^{\alpha}(L) f_{nk}^{\alpha\beta}(D) \mathcal{D}_{kl}^{\beta}(R^{-1}). \quad (14)$$

In the physical case $n = 3$, we have obviously the standard form of $\text{SO}(3, \mathbb{R})$ -Casimirs:

$$\mathbf{C}_{\text{SO}(V, g)}(2) = \mathbf{S}_1^2 + \mathbf{S}_2^2 + \mathbf{S}_3^2 = \hat{\mathbf{r}}_1^2 + \hat{\mathbf{r}}_2^2 + \hat{\mathbf{r}}_3^2 = \mathbf{C}_{\text{SO}(3, \mathbb{R})}(2),$$

$$\mathbf{C}_{\text{SO}(U, \eta)}(2) = \mathbf{V}_1^2 + \mathbf{V}_2^2 + \mathbf{V}_3^2 = \hat{\mathbf{t}}_1^2 + \hat{\mathbf{t}}_2^2 + \hat{\mathbf{t}}_3^2 = \mathbf{C}_{\text{SO}(3, \mathbb{R})}(2).$$

Our expansions for wave functions are then described in terms of well-known expressions found by Wigner, and, of course, the family of rotational Casimirs begins and terminates on $p = 2$.

Obviously, for $n = 3$, Ω is the set of non-negative integer, α, β are traditionally denoted by symbols like $s, j = 0, 1, 2, \dots$, etc., $N(s) = 2s + 1$, $N(j) = 2j + 1$, and the indices (m, n) , (k, l) are considered as jumping by 1, respectively, from $-s$ to s and from $-j$ to j ; here the tradition is too strong to respect the formal logical conventions. Thus, the expansion (12) is written according to the mentioned conventions:

$$\Psi(\varphi) = \Psi(L, D, R) = \sum_{s, j=0}^{\infty} \sum_{m, n=-s}^s \sum_{k, l=-j}^j \mathcal{D}_{mn}^s(L) f_{nk}^{sj}(D) \mathcal{D}_{kl}^j(R^{-1}). \quad (15)$$

Similarly, the reduced amplitudes (14) are written as:

$$\Psi(\varphi) = \Psi_{ml}^{sj}(L, D, R) = \sum_{n=-s}^s \sum_{k=-j}^j \mathcal{D}_{mn}^s(L) f_{nk}^{sj}(D) \mathcal{D}_{kl}^j(R^{-1}). \quad (16)$$

Here \mathcal{D}^s are celebrated Wigner matrices of $(2s + 1)$ -dimensional irreducible representations of the three-dimensional rotation group. They are well-known

special functions of mathematical physics and may be assumed to be something in principle standard and well-know.

Obviously, the amplitudes Ψ_{ml}^{sj} are eigenfunctions of rotational Casimir invariants, i.e., essentially angular momentum and vorticity:

$$\|\mathbf{S}\|^2 \Psi_{ml}^{sj} = \|\hat{\mathbf{r}}\|^2 \Psi_{ml}^{sj} = \hbar^2 s(s+1) \Psi_{ml}^{sj}, \quad \|\mathbf{V}\|^2 \Psi_{ml}^{sj} = \|\hat{\mathbf{t}}\|^2 \Psi_{ml}^{sj} = \hbar^2 j(j+1) \Psi_{ml}^{sj},$$

where, let us remind, in three dimensions we have $\|\mathbf{S}\|^2 = \mathbf{S}_1^2 + \mathbf{S}_2^2 + \mathbf{S}_3^2$, $\|\mathbf{V}\|^2 = \mathbf{V}_1^2 + \mathbf{V}_2^2 + \mathbf{V}_3^2$, and similarly for $\hat{\mathbf{r}}$, $\hat{\mathbf{t}}$. According to tradition, one uses such a basis that Ψ_{ml}^{sj} are also eigenfunctions of the third components of rotational generators,

$$\mathbf{S}_3 \Psi_{ml}^{sj} = \hbar m \Psi_{ml}^{sj}, \quad \mathbf{V}_3 \Psi_{ml}^{sj} = \hbar l \Psi_{ml}^{sj}.$$

And, obviously, when the values n, k in the superposition (16) are kept fixed and we retain only the corresponding single term, for the resulting Ψ we have

$$\hat{\mathbf{r}}_3 \Psi_{nk}^{sj} = \hbar n \Psi_{nk}^{sj}, \quad \hat{\mathbf{t}}_3 \Psi_{nk}^{sj} = \hbar k \Psi_{nk}^{sj}.$$

14 Representation matrices

In this way one is dealing with quantum states of well-definite values of magnitudes and third components of the angular momentum and vorticity. For the general n , the amplitudes $\Psi_{ml}^{\alpha\beta}$ have, of course, the well-definite values $(\hbar/i)^p C(\alpha, p)$, $(\hbar/i)^p C(\beta, p)$ of the Casimirs (13). And now it will be convenient to return for a while (at least in a formal way) to the general case of dimension n .

Let us again use the exponential formulas (9) for the elements of $W(\omega) \in \text{SO}(V, g)$, $W(\omega) \in \text{SO}(U, \eta)$, and just their simply numerical counterparts in $\text{SO}(n, \mathbb{R})$,

$$W(\omega) = \exp \left(\frac{1}{2} \omega^a_b E^b_a \right),$$

where the basic matrices E^b_a are simply given by $(E^b_a)^c_d = \delta^b_d \delta^c_a - \delta^{bc} \delta_{ad}$ (just simply the numerical counterpart of (9) showing that one works just in \mathbb{R}^n and $\text{SO}(n, \mathbb{R})'$ not in V, U , $\text{SO}(V, g)$, $\text{SO}(U, \eta)$ basis-identified with the previous ones). And from now on let us again decide to work in purely analytical matrix form using orthonormal coordinates in V, U and identifying them with \mathbb{R}^n . Representation matrices \mathcal{D}^α are given by the following expresion:

$$\mathcal{D}^\alpha(\omega) = \exp \left(\frac{1}{2} \omega^a_b M^{\alpha b}_a \right),$$

where $N(\alpha) \times N(\alpha)$ anti-hermitian matrices $M^{\alpha b}_a$ form irreducible representations of the Lie algebra $\text{SO}(n, \mathbb{R})'$, thus, their commutation rules are identical with those for E^b_a .

Remark: For any $\alpha \in \Omega$ and for any pair of indices b, a , $M^{\alpha b}_a$ are just matrices not (b, a) -matrix elements of some M^α ; let us notice in this connection

that $a, b = \overline{1, n}$, whereas any $M^{\alpha b}_a$ is an $N(\alpha) \times N(\alpha)$ -matrix. Obviously, when dealing with matrices $\mathcal{D}^\alpha(L)$, $\mathcal{D}^\beta(R)$, we must specialize the redundant "coordinates" ω^a_b to the ones parameterizing respectively the L - and R -terms of the two-polar splitting, writing, e.g.,

$$\mathcal{D}^\alpha(L(l)) = \exp\left(\frac{1}{2}l^a{}_b M^{\alpha b}_a\right), \quad \mathcal{D}^\beta(R(r)) = \exp\left(\frac{1}{2}r^a{}_b M^{\beta b}_a\right).$$

For example, in three dimensions, where the pseudovector \bar{k} may be used instead of the tensor ω^b_a , i.e., $\mathcal{D}^s(W(\bar{k})) = \exp(k^a M^s_a)$, we should write that $\mathcal{D}^s(L(\bar{l})) = \exp(l^a M^s_a)$, $\mathcal{D}^j(R(\bar{r})) = \exp(r^a M^j_a)$, where M^s_a (s being non-negative integers and $a = 1, 2, 3$) are basic $(2s+1) \times (2s+1)$, thus, odd-dimensional, anti-hermitian matrices representing in an irreducible way the Lie algebra $\text{SO}(3, \mathbb{R})'$. Therefore, $[M^s_a, M^s_b] = -\epsilon_{ab}{}^c M^s_c$, and it is impossible to reduce simultaneously all M^s_a to the block form. The apparently impossible even dimension $(2s+1)$ of M^s_a , thus, positive half-integer s will be an important point of our further analysis because $\text{SO}(3, \mathbb{R})'$ (just as any $\text{SO}(n, \mathbb{R})'$, $n \geq 3$) admits even-dimensional representations corresponding to the half-integer angular momentum, both for rigid and homogeneously deformable bodies.

15 Algebraic form of equations

Let us introduce Hermitian matrices $S^{\alpha a}_b = (\hbar/i)M^{\alpha a}_b$, thus, for $n = 3$, $S^j_a = (\hbar/i)M^j_a$, and

$$\frac{1}{i\hbar}[S^j_a, S^j_b] = \epsilon_{ab}{}^c S^j_c.$$

These are standard well-known matrices, possible to be determined in purely algebraic terms, basing only on the commutation relations [8]. And it was just a surprise that there exist even-dimensional irreducible representations, experimentally compatible with the half-integer internal angular momentum spin. The $(2j+1) \times (2j+1)$ matrices S^j provide the quantum description of the angular momentum with the quantized magnitude $\hbar^2 j(j+1)$; j being a non-negative integer, or also a positive half-integer in the theory of fermionic objects.

The representation property of \mathcal{D}^α , i.e., $\mathcal{D}^\alpha(R_1 R_2) = \mathcal{D}^\alpha(R_1) \mathcal{D}^\alpha(R_2)$, together with the definition of generators (8), (10) imply that certain obvious relationships which enable one to replace some differential operations and equations by algebraic ones. Namely, it is clear from the above formulas that

$$\begin{aligned} \frac{\hbar}{i} \mathbf{\Lambda}^i_j(L) \mathcal{D}^\alpha(L) &= S^{\alpha i}_j \mathcal{D}^\alpha(L), & \frac{\hbar}{i} \mathbf{\Lambda}^A_B(R) \mathcal{D}^\beta(R) &= \mathcal{D}^\beta(R) S^{\beta A}_B, \\ \frac{\hbar}{i} \mathbf{\Upsilon}^a_b(L) \mathcal{D}^\alpha(L) &= \mathcal{D}^\alpha(L) S^{\alpha a}_b, & \frac{\hbar}{i} \mathbf{\Upsilon}^a_b(R) \mathcal{D}^\beta(R) &= S^{\beta a}_b \mathcal{D}^\beta(R); \end{aligned}$$

expressions on the right-hand side meant, obviously, in the sense of the matrix multiplication.

In other words, $\mathbf{S}^i_j \Psi^{\alpha\beta} = S^{\alpha i}_j \Psi^{\alpha\beta}$ and $\mathbf{V}^A_B \Psi^{\alpha\beta} = \Psi^{\alpha\beta} S^{\beta A}_B$, where $\Psi^{\alpha\beta}$ is an abbreviation for the $N(\alpha) \times N(\beta)$ matrices $\left[\Psi^{\alpha\beta}_{ml} \right]$ in (14), $m = \overline{1, N(\alpha)}$, $l = \overline{1, N(\beta)}$. Obviously, everything is formally correct because $S^{\alpha i}_j$, $S^{\beta A}_B$ are, respectively, $N(\alpha) \times N(\alpha)$ - and $N(\beta) \times N(\beta)$ -matrices. Let us stress once again that the indices (i, j) , (A, B) label basic matrices within their sets; they do not refer to matrix elements.

From now on it will be convenient to write also (14), (16) in matrix terms,

$$\Psi^{\alpha\beta}(L, D, R) = \mathcal{D}^\alpha(L) f^{\alpha\beta}(D) \mathcal{D}^\beta(R^{-1});$$

obviously, the reduced amplitude $f^{\alpha\beta}(D)$ is an $N(\alpha) \times N(\beta)$ -matrix depending only on deformation invariants $D_{aa} = Q^a = \exp(q^a)$.

Similarly, $\hat{\mathbf{r}}^a_b$ and $\hat{\mathbf{t}}^a_b$ act on $\Psi^{\alpha\beta}$ as follows:

$$\hat{\mathbf{r}}^a_b \Psi^{\alpha\beta} = \mathcal{D}^\alpha(L) S^{\alpha a}_b f^{\alpha\beta}(D) \mathcal{D}^\beta(R^{-1}),$$

$$\hat{\mathbf{t}}^a_b \Psi^{\alpha\beta} = \mathcal{D}^\alpha(L) f^{\alpha\beta}(D) S^{\beta a}_b \mathcal{D}^\beta(R^{-1}).$$

Therefore, this action reduces simply to the action on the reduced amplitude $f^{\alpha\beta}$ only. It will be convenient to denote it as follows: $\overrightarrow{S}^{\alpha a}_b f^{\alpha\beta} := S^{\alpha a}_b f^{\alpha\beta}$, $\overleftarrow{S}^{\beta a}_b f^{\alpha\beta} := f^{\alpha\beta}(D) S^{\beta a}_b$. By assumption, the representations \mathcal{D}^α of $\text{SO}(n, \mathbb{R})$ are irreducible, therefore, the matrices $C^\alpha(p) = S^{\alpha a}_b S^{\alpha b}_c \dots S^{\alpha u}_w S^{\alpha w}_a$ (with p factors) are proportional to the $N(\alpha) \times N(\alpha)$ identity matrices,

$$C^\alpha(p) = \left(\frac{\hbar}{i} \right)^p C(\alpha, p) I_{N(\alpha)}, \quad (17)$$

where the numbers $C(\alpha, p)$ are eigenvalues of the corresponding Casimir operators built of the generators of the left and right regular translations on $\text{SO}(n, \mathbb{R})$, e.g., $\mathbf{C}_{\text{SO}(n, \mathbb{R})}(p) = \mathbf{\Lambda}^a_b \mathbf{\Lambda}^b_c \dots \mathbf{\Lambda}^u_w \mathbf{\Lambda}^w_a$ (with p factors).

So, finally, let us summarize the corresponding formulas for the physical case $n = 3$,

$$\|\mathbf{S}\|^2 \Psi^{sj} = \|\hat{\mathbf{r}}\|^2 \Psi^{sj} = \hbar^2 s(s+1) \Psi^{sj}, \quad \|\mathbf{V}\|^2 \Psi^{sj} = \|\hat{\mathbf{t}}\|^2 \Psi^{sj} = \hbar^2 j(j+1) \Psi^{sj},$$

$$\mathbf{S}_a \Psi^{sj} = S^s_a \Psi^{sj}, \quad \mathbf{V}_a \Psi^{sj} = \Psi^{sj} S^j_a,$$

in particular, in the standard representation, $\mathbf{S}_3 \Psi^{sj}_{ml} = \hbar m \Psi^{sj}_{ml}$, $\mathbf{V}_3 \Psi^{sj}_{ml} = \hbar l \Psi^{sj}_{ml}$. And just as for the general dimension value n , a little more complicated action of $\hat{\mathbf{r}}_a$, $\hat{\mathbf{t}}_a$ resulting in affecting the reduced $f(D)$ -amplitudes,

$$\hat{\mathbf{r}}_a : f^{sj} \mapsto S^s_a f^{sj} = \overrightarrow{S}^s_a f^{sj}, \quad \hat{\mathbf{t}}_a : f^{sj} \mapsto f^{sj} S^j_a = \overleftarrow{S}^j_a f^{sj}.$$

In the standard representation we have $\hat{\mathbf{r}}_3 : \left[f^{sj}_{ml} \right] \mapsto \left[\hbar m f^{sj}_{ml} \right]$, $\hat{\mathbf{t}}_3 : \left[f^{sj}_{ml} \right] \mapsto \left[\hbar l f^{sj}_{ml} \right]$.

16 Half-integer values of spin

And now we are ready to return to the problem of covering spaces and half-integer quantized angular momentum of rigid and deformable bodies. The problem of half-integer spin appeared in quantum mechanics due to experimental data concerning radiation spectra of atoms and molecules. Later on some theoretical work gave an evidence of the existence of even-dimensional representations of the Lie algebra $\text{SO}(3, \mathbb{R})'$. Their exponentiation does not lead to representations of $\text{SO}(3, \mathbb{R})$ but to representations of its universal covering group $\text{SU}(2)$, roughly speaking to the double-valued representations of $\text{SO}(3, \mathbb{R})$; in a sense to its projective representations.

As mentioned, there are some arguments that, in contrast to the current views, it need not be always the case that the wave amplitudes must be one-valued functions on the configuration space. In certain situations, when the homotopy group is finite, it seems to be sufficient that they are correctly defined on the universal covering manifold of the configuration space. A typical example is rigid body mechanics [1, 2, 3] and the mechanics of affinely-rigid bodies.

Let us begin with the general n -dimensional case, $n \geq 3$. The configuration space of the rigid body in n dimensions may be identified analytically with the special orthogonal group $\text{SO}(n, \mathbb{R})$. For $n \geq 3$ the universal covering group $\text{Spin}(n)$ is doubly-connected, and the corresponding canonical projection $\tau : \text{Spin}(n) \rightarrow \text{SO}(n, \mathbb{R})$ is $2 : 1$. The special case $n = 2$ is completely different, because then the homotopy group is \mathbb{Z} just as in the covering of the circle $\text{SO}(2, \mathbb{R}) \simeq \text{U}(1)$ by \mathbb{R} . Therefore, in this case it does not seem possible to admit multi-valued wave functions, i.e., ones defined on \mathbb{R} . For $n = 3$ the covering group $\text{Spin}(3)$ is isomorphic with the group of special (determinant-one) unitary matrices $\text{SU}(2)$. For any $u \in \text{SU}(2)$ the matrices $\pm u \in \text{SU}(2)$ project under τ onto the same element of $\text{SO}(3, \mathbb{R})$. Therefore, $\text{Ker } \tau = \tau^{-1}(I_3) = \{I_2, -I_2\}$, i.e., the kernel consists of the unit 2×2 matrix I_2 and $-I_2$. Lie algebra $\text{SU}(2)'$ consists of anti-hermitian traceless complex matrices, i.e., such ones that $\alpha^+ = \bar{\alpha}^T = -\alpha$, $\text{Tr } \alpha = 0$. The most convenient choice of basis, commonly used in geometry and physics, is the following system: $e_a = (1/2i)\sigma_a$, $a = 1, 2, 3$, where σ_a are Pauli matrices:

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

The basis $\{e_a\}$ exactly corresponds to the basis $\{E_a\}$ of $\text{SO}(3, \mathbb{R})'$, i.e., $[e_a, e_b] = \epsilon_{ab}^c e_c$. Canonical coordinates of the first kind are given by components of rotation vector $\bar{k} \in \mathbb{R}^3$,

$$u(\bar{k}) = \exp(k^a e_a) = \cos \frac{k}{2} I_2 - \frac{k^a}{k} \sin \frac{k}{2} i \sigma_a; \quad (18)$$

often one uses the relativistic convention σ_0 for I_2 . This parameterization exactly corresponds to the usual rotation vector $\bar{k} \in \mathbb{R}^3$ in $\text{SO}(3, \mathbb{R})'$, thus, $\tau(u(\bar{k})) = R(\bar{k})$ if $k \leq \pi$. The main difference is that k , the magnitude of \bar{k} , runs over the doubled range $[0, 2\pi]$. Parameterization is singular in the sense

that all points of the limiting sphere $k = 2\pi$ in \mathbb{R}^3 represent the same point of $\text{SU}(2)$, namely, $-I = u(2\pi \cdot \bar{n})$ for any versor $\bar{n} \in \mathbb{R}^3$, $(\bar{n} \cdot \bar{n}) = 1$. All points of \mathbb{R}^3 in the interior of the ball $k < 2\pi$ represent uniquely elements of $\text{SU}(2)$; in particular, unlike the situation in $\text{SO}(3, \mathbb{R})$ there is no antipodal identification for $k = \pi$, i.e., $u(\pi \cdot \bar{n}) \neq u(-\pi \cdot \bar{n})$. Obviously,

$$\tau^{-1}(R(\bar{k})) = \{u(\bar{k}), -u(\bar{k})\} = \left\{u(\bar{k}), u\left(\left(1 - \frac{2\pi}{k}\right)\bar{k}\right)\right\}.$$

The epimorphism $\tau : \text{SU}(2) \rightarrow \text{SO}(3, \mathbb{R})$ is given by the assignment: $\text{SU}(2) \ni v \mapsto R \in \text{SO}(3, \mathbb{R})$, where $vu(\bar{k})v^{-1} = u(R\bar{k})$. For any non-negative integer or positive half-integer $s = 0, 1/2, 1, 3/2, \dots$ ($s = j/2$, $j = 0, 1, 2, 3, \dots$) the Lie algebra $\text{SU}(2)'$ does possess an irreducible representation of dimension $(2s+1)$ (thus, all naturals admitted, not only the even ones) in terms of anti-hermitian matrices M^s , the basic ones M^s_a , $a = 1, 2, 3$, chosen so as to satisfy $[M^s_a, M^s_b] = -\epsilon_{ab}^c M^s_c$. Obviously, the corresponding s -angular momentum matrices, $S^s_a := (\hbar/i)M^s_a$, are Hermitian and do commute: $(1/i\hbar)[S^s_a, S^s_b] = \epsilon_{ab}^c S^s_c$. Exponentiation

$$\mathcal{D}^s(u(\bar{k})) = \mathcal{D}^s(\exp(k^a e_a)) := \exp(k^a M^s_a) = \exp\left(\frac{i}{\hbar} k^a S^s_a\right)$$

leads to unitary irreducible $(2s+1)$ -dimensional representations of $\text{SU}(2)$. The matrices M^s_a , as mentioned, may be found on the purely algebraic basis of commutation relations [8], thus, for $n = 3$ they are explicitly known and standard. And so are \mathcal{D}^s , or more precisely their matrix elements as the special functions on $\text{SO}(3, \mathbb{R})$. This, by the way, is not the only possible method. Another one is solving of differential equations on the group manifold, or symmetrized Kronecker products of the basic representation of $\text{SU}(2)$ by itself.

For non-negative integers s , i.e., for the odd values of $N(s) = (2s+1)$, \mathcal{D}^s do not distinguish elements $\pm u \in \text{SU}(2)$ τ -projecting onto the same elements $R \in \text{SO}(3, \mathbb{R})$, so, as a matter of fact, they are representations of $\text{SO}(3, \mathbb{R})$ (more precisely, they are τ -pull-backs of $\text{SO}(3, \mathbb{R})$ -representations to $\text{SU}(2)$), just the previously discussed \mathcal{D}^s ,

$$\mathcal{D}^s(u) = \mathcal{D}^s(-u).$$

For the positive half-integers s , i.e., for the even values of $N(s) = (2s+1)$, \mathcal{D}^s differ in sign at $u, -u \in \text{SU}(2)$,

$$\mathcal{D}^s(u) = -\mathcal{D}^s(-u),$$

thus, they are non-projectable to $\text{SO}(3, \mathbb{R})$. But for any fixed s , the squared moduli of the matrix elements or those of their linear combinations are pull-backs from $\text{SO}(3, \mathbb{R})$, so the probabilistic interpretation of $\bar{\Psi}\Psi$ is not violated. The same holds when we superpose matrix elements of various \mathcal{D}^s , \mathcal{D}^j but with the same parity of $(2s+1)$, $(2j+1)$, i.e., with the same "half-nesses" of s, j . But, in general, the probabilistic interpretation of $\bar{\Psi}\Psi$ is violated when different "half-nesses" of s, j are superposed. This is a toy model of the superselection

between "fermionic" and "bosonic" situations. As we shall see, in a much more drastic form the problem appears in quantum mechanics of affinely-rigid bodies.

Having in view physical applications we do not consider the general case with $n > 3$, thus, our $\text{Spin}(n)$ will be $\text{Spin}(3) \simeq \text{SU}(2)$. The planar problems $n = 2$ are of some physical relevance and will be briefly reviewed. However, the possibility of the half-integer spin does not appear then; at the same time, some other problems difficult for $n = 3$ become drastically simplified, just trivialized, for $n = 2$.

17 Affine spinors and polar decompositions

The configuration spaces of affinely-rigid body, i.e., roughly speaking (if translational motion is neglected) $\text{GL}^+(n, \mathbb{R})$, $\text{SL}(n, \mathbb{R})$, are also doubly-connected, and the problem of physically admissible two-valued wave functions also appears here. There is, however, some difficulty, namely, the intriguing and interesting fact that the universal covering groups $\text{GL}^+(n, \mathbb{R})$, $\overline{\text{SL}(n, \mathbb{R})}$ are nonlinear, i.e., they do not possess faithful realizations in terms of finite matrices. This, by the way, was a reason for plenty of misunderstandings and vast time in field theory and quantum mechanics [6, 7]. The fact was known long ago to mathematicians, like, e.g., E. Cartan, but was forgotten and exotic for physicists. The nonlinearity of the mentioned coverings implies, in particular, that affine spinors (half-objects) must be either infinite-dimensional or ruled by nonlinear realizations of $\overline{\text{GL}^+(n, \mathbb{R})}$, $\overline{\text{SL}(n, \mathbb{R})}$ as abstract groups constructed with the help of loops in $\text{GL}^+(n, \mathbb{R})$, $\text{SL}(n, \mathbb{R})$.

However, in quantum mechanics of affinely-rigid bodies the construction of multi-valued wave functions may be analytically overcome with the use of polar and two-polar splittings. Let us begin from the first one,

$$\varphi = UA = BU = (UAU^{-1})U,$$

where $U \in \text{SO}(n, \mathbb{R})$, and A, B are symmetric and positively definite (and in the case of $\text{SL}(n, \mathbb{R})$ their determinants equal one). The splitting is unique and, because of this, $\text{GL}^+(n, \mathbb{R})$ as a manifold (but not as a group) may be identified with the Cartesian products $\text{SO}(n, \mathbb{R}) \times \text{Sym}^+(n, \mathbb{R})$ or $\text{Sym}^+(n, \mathbb{R}) \times \text{SO}(n, \mathbb{R})$. The manifold $\text{Sym}^+(n, \mathbb{R})$ is diffeomorphic with $\mathbb{R}^{n(n+1)/2}$ (\mathbb{R}^6 if $n = 3$), therefore, the covering manifold may be identified with $\text{Spin}(n) \times \text{Sym}^+(n, \mathbb{R})$ or $\text{Sym}^+(n, \mathbb{R}) \times \text{Spin}(n)$. In the physical case $n = 3$, these splittings become $\text{SU}(2) \times \text{Sym}^+(3, \mathbb{R}) \simeq \text{SU}(2) \times \mathbb{R}^6$ or alternatively $\text{Sym}^+(3, \mathbb{R}) \times \text{SU}(2) \simeq \mathbb{R}^6 \times \text{SU}(2)$. Topological non-triviality is absorbed here by the factor $\text{SO}(3, \mathbb{R})$ (in general by $\text{SO}(n, \mathbb{R})$) and covered by $\text{SU}(2)$ (in general by $\text{Spin}(n)$). Therefore, the admissible multi-valued wave functions may be expanded as follows:

$$\Psi(u, A) = \sum_s \sum_{m=-s}^s \sum_{k=-s}^s C^s_{mk}(A) \mathcal{D}^s_{mk}(u),$$

where s are non-negative integers or positive half-integers, and the summation over m, k is performed in steps by one, \mathcal{D}^s are matrices of irreducible unitary

representations of $SU(2)$, and (very important!) only half-integer or integer values of s may appear in a given expansion if $\bar{\Psi}\Psi$ is to be one-valued on $GL^+(3, \mathbb{R})$, or, more precisely, if it is to be a pull-back from $GL^+(3, \mathbb{R})$ to $\overline{GL^+(3, \mathbb{R})}$. Therefore, in any admissible Ψ , $C^s_{mk} = 0$ either for all non-negative integer or for all positive half-integer s . To be completely rigorous, we would have to write

$$\Psi(u, A) = \sum_{\sigma=1}^{\infty} \sum_{\mu=0}^{\sigma} \sum_{\kappa=0}^{\sigma} C^{\frac{\sigma}{2}}_{(-\frac{\sigma}{2}+\mu), (-\frac{\sigma}{2}+\kappa)}(A) \mathcal{D}^{\frac{\sigma}{2}}_{(-\frac{\sigma}{2}+\mu), (-\frac{\sigma}{2}+\kappa)}(u)$$

for half-integer spin ("fermionic") situations or, respectively,

$$\Psi(u, A) = \sum_{s=0}^{\infty} \sum_{\mu=0}^{2s} \sum_{\kappa=0}^{2s} C^s_{(-s+\mu), (-s+\kappa)}(A) \mathcal{D}^s_{(-s+\mu), (-s+\kappa)}(u)$$

for integer spin ("bosonic") situations. These formulas are valid without any provisos, with summation over all indices meant in steps by one. If $\bar{\Psi}\Psi$ is to be one-valued probability distribution, then the superposing between indicated subspaces of function series is forbidden (a kind of superselection rule), and the admissible Hamiltonians must exclude any transitions between them; otherwise they are not well-defined on $L^2(GL^+(n, \mathbb{R}))$.

It was said that the two-polar decomposition is maximally effective in problems on which we concentrate. Let us now describe the covering manifold $\overline{GL^+(n, \mathbb{R})}$ and the corresponding two-valued wave functions on $GL^+(n, \mathbb{R})$ in terms of the two-polar splitting. The non-uniqueness of the two-polar splitting of $GL^+(n, \mathbb{R})$ was described briefly at the beginning of section 6 of Part I [20]. Certain modifications are necessary when using this splitting for describing the covering $\overline{GL^+(n, \mathbb{R})}$.

The elements of $GL^+(n, \mathbb{R})$ were represented by the triplets $(L, D, R) \in SO(n, \mathbb{R}) \times \mathbb{R}^n \times SO(n, \mathbb{R})$ taken modulo certain identifications resulting from the fact that it was just the product $\varphi = LDR^{-1}$ not (L, D, R) itself that was a true configuration. Now, when describing $\overline{GL^+(n, \mathbb{R})}$, we must start from the triplets $(l, D, r) \in Spin(n) \times \mathbb{R}^n \times Spin(n)$, i.e., in the physical three-dimensional case $(l, D, r) \in SU(2) \times \mathbb{R}^3 \times SU(2)$. In this last case, l and r will be analytically described by the extended rotation vectors $\bar{l}, \bar{r} \in \mathbb{R}^3$ in the sense of (18) with \bar{k} replaced respectively by \bar{l}, \bar{r} . Similarly, D is analytically represented by the variables $q^a = \ln D_{aa}$, and the dilatational degree of freedom by the centre $q = (q^1 + q^2 + q^3)/3$. As above, $\tau : Spin(n) \rightarrow SO(n, \mathbb{R})$ denotes the canonical projection (2 : 1 epimorphism). $\overline{K^+} := \tau^{-1}(K^+)$ is a $2n \cdot n!$ -element subgroup of $Spin(n)$; the group $K^+ \subset SO(n, \mathbb{R})$ itself was defined in section 6 of Part I [20]. The manifold $M^{(n)}$ introduced also there is covered by $\overline{M^{(n)}}$, i.e., the subset of such triplets $(l; q^1, \dots, q^n; r) \in Spin(n) \times \mathbb{R}^n \times Spin(n)$ that all q^i 's are pairwise distinct. The subgroup $\overline{K^+}$ induces on $\overline{M^{(n)}}$ the transformation group $\overline{H^{(n)}}$ action of which is given by the following rule: $(l; q^1, \dots, q^n; r) \mapsto (lu; q^{\pi_{\tau(u)}(1)}, \dots, q^n; ru)$, where $u \in \overline{K^+} \subset Spin(n)$ ($SU(2)$ if $n = 3$).

The corresponding generic part of $\overline{\text{GL}^+(n, \mathbb{R})}$ (non-degenerate deformation tensors) is obtained as a quotient subset under the $\overline{H^{(n)}}$ -action, i.e., $Q^{(n)} \simeq \overline{M^{(n)}}/\overline{H^{(n)}}$. Situation becomes more complicated when some q^a 's coincide, i.e., when the spectra of deformation tensors are degenerate. Let the symbols $Q^{(k;p_1, \dots, p_k)} = \text{GL}^{+(k;p_1, \dots, p_k)}$, p_σ , $M^{(k;p_1, \dots, p_k)}$, k , and $H^{(k;p_1, \dots, p_k)}$ have the same meaning as in the beginning of section 6 of Part I [20] where the two-polar splitting non-uniqueness was described. To describe the half-integer angular momentum, we must take the manifold $\overline{M^{(k;p_1, \dots, p_k)}}$ consisting of triplets $(l; q^1, \dots, q^n; r)$, where $l, r \in \text{Spin}(n)$ and the system (q^1, \dots, q^n) is degenerate as above. Let $\overline{H^{(k;p_1, \dots, p_k)}} \subset \text{Spin}(n)$ denote the subgroup $\tau^{-1}(H^{(k;p_1, \dots, p_k)})$. The corresponding manifolds of degenerate configurations are given by the quotient subsets $\overline{M^{(k;p_1, \dots, p_k)}}/\overline{H^{(k;p_1, \dots, p_k)}}$ in the sense of the following action: $(l; q^1, \dots, q^n; r) \mapsto (lu; q^{\pi_{\tau(u)}(1)}, \dots, q^n; ru)$; obviously, u runs over $\overline{H^{(k;p_1, \dots, p_k)}}$.

The admissibly multi-valued wave functions on $\text{GL}^+(n, \mathbb{R})$, i.e., the ones one-valued on $\overline{\text{GL}^+(n, \mathbb{R})}$, are represented by complex amplitudes on $\text{Spin}(n) \times \mathbb{R}^n \times \text{Spin}(n)$ which are invariant under the above actions of $\overline{H^{(k;p_1, \dots, p_k)}}$, i.e., are projectable onto the resulting quotients $\overline{M^{(k;p_1, \dots, p_k)}}/\overline{H^{(k;p_1, \dots, p_k)}}$.

18 Three-dimensional physical case

Let us now concentrate on the special case $n = 3$, both the practically important one and at the same time reducible in a sense to the classical Wigner results [14, 22, 23].

All the former expressions concerning function series, eigenequations, etc. remain generally true with the following changes: half-integer quantum numbers s, j, m, k, l, n , etc. are admissible, and certain new complications appear concerning the non-distinguishability of triplets $(l; q^1, q^2, q^3; r)$ by wave functions representants. In particular, some correlation appears between "half-nesses" of the quantum numbers s, j (spin and vorticity) in physically acceptable function series. Obviously, this is based on the assumption (true or not?) that the wave functions Ψ may be multi-valued, but their moduli $|\Psi|$ must be one-valued in accordance with the statistical interpretation of $\overline{\Psi}\Psi$.

It is known that $\mathcal{D}^j(u) = \pm \mathcal{D}^j(-u)$, $u \in \text{SU}(2)$, depending, respectively, on whether j is integer or half-integer. Therefore, the expansions (15), (16) remain valid for half-integer spin and vorticity, thus, within the framework $\text{SU}(2) \times \mathbb{R}^3 \times \text{SU}(2)$ provided that some care is taken what concerns the superposition structure, more precisely, the (s, j) correlation. So, formally, we can rewrite (15), (16) as follows:

$$\Psi(u, D, v) = \sum_{s,j} \sum_{m,n=-s}^s \sum_{k,l=-j}^j \mathcal{D}_{mn}^s(u) f_{nk}^{sj}(D) \mathcal{D}_{kl}^j(v^{-1}), \quad (19)$$

$$\Psi_{ml}^{sj}(u, D, v) = \sum_{n=-s}^s \sum_{k=-j}^j \mathcal{D}_{mn}^s(u) f_{nk}^{sj}(D) \mathcal{D}_{kl}^j(v^{-1}), \quad (20)$$

with the following descriptive comments. Summation over (s, j) in (19) or the choice of particular (s, j) in (20) is extended over non-negative integers or positive half-integers, but in such a way that either (s, j) are simultaneously integers or simultaneously half-integers. In other words, $f^{sj}(q^1, q^2, q^3) \equiv 0$ if the number $(j - s)$ is half-integer, i.e., always the summation will be extended over such pairs (s, j) in (19) or the values of (s, j) will be chosen in (20) in such a way that $(j - s)$ will be an integer number. The quantum numbers (m, n) , (k, l) in (19) run over the ranges from $-s$ to s and from $-j$ to j in integer jumps.

Just as for integer pairs (s, j) , we will use $(2s + 1) \times (2j + 1)$ rectangular matrices $\Psi^{sj} = [\Psi_{ml}^{sj}]$, $f^{sj} = [f_{nk}^{sj}]$, where

$$\Psi^{sj}(u; q^1, q^2, q^3; v) = \mathcal{D}^s(u) f^{sj}(q^1, q^2, q^3) \mathcal{D}^j(v^{-1}).$$

As mentioned, f^{sj} vanishes identically as a function of q^a 's when $(j - s)$ is half-integer. The matrix elements of Ψ^{sj} with integer values of $(j - s)$ may be arbitrarily superposed, and this correlation is a necessary condition if Ψ , Ψ^{sj} are to be well-defined on $\overline{\text{GL}(3, \mathbb{R})}$ not only on the auxiliary manifold $\text{SU}(2) \times \mathbb{R}^3 \times \text{SU}(2)$. If $\overline{\Psi}\Psi$ is to be projectable onto $\text{GL}(3, \mathbb{R})$ (statistical interpretation), then we may superpose only terms with half-integer (s, j) or integer (s, j) separately.

One is used to avoid in mathematical texts the descriptive literature-like comments as above, however, sometimes the purely formula-based presentation becomes more obscure. It is just the case here, especially when we wish to retain the traditional notation used in the theory of angular momentum. So, for example, avoiding words following the formulas (19), (20) would be panished by the following rather obscure expressions:

$$\begin{aligned} \Psi(u, D, v) &= \Psi_1(u, D, v) + \Psi_2(u, D, v) = \\ &= \sum_{\sigma, \iota=1}^{\infty} \sum_{\mu, \nu=0}^{\sigma} \sum_{\kappa, \lambda=0}^{\iota} \mathcal{D}_{(-\frac{\sigma}{2}+\mu), (-\frac{\sigma}{2}+\nu)}^{\frac{\sigma}{2}}(u) f_{(-\frac{\sigma}{2}+\mu), (-\frac{\iota}{2}+\kappa)}^{\frac{\sigma}{2}, \frac{\iota}{2}}(D) \mathcal{D}_{(-\frac{\iota}{2}+\kappa), (-\frac{\iota}{2}+\lambda)}^{\frac{\iota}{2}}(v^{-1}) \\ &+ \sum_{s, j=0}^{\infty} \sum_{\mu, \nu=0}^{2s} \sum_{\kappa, \lambda=0}^{2j} \mathcal{D}_{(-s+\mu), (-s+\nu)}^s(u) f_{(-s+\mu), (-j+\kappa)}^{sj}(D) \mathcal{D}_{(-j+\kappa), (-j+\lambda)}^j(v^{-1}). \end{aligned}$$

The first term Ψ_1 contains contributions with half-integer spin and vorticity (simultaneously), the second one Ψ_2 involves only integer quantized values of both. And this fact again means that Ψ is well-defined on $\overline{\text{GL}(3, \mathbb{R})}$ not only on $\text{SU}(2) \times \mathbb{R}^3 \times \text{SU}(2)$. But if $\overline{\Psi}\Psi$ is to be well-defined on $\text{GL}(3, \mathbb{R})$ itself, then only Ψ_1, Ψ_2 -terms are separately admissible without being superposed.

19 Reduction to Cartan subgroup

Matrix elements of irreducible representations have important well-investigated properties which enable one to algebraize a good deal of differential equations problems and to perform an effective reduction of the quantum dynamics.

Roughly speaking, this is reduction to the Cartan subgroup of $GL(n, \mathbb{R})$, i.e., to its maximal Abelian subgroup. This is just the group of diagonal matrices, i.e., degrees of freedom parameterized by deformation invariants q^1, \dots, q^n . This reduction from n^2 to n degrees of freedom is possible for geodetic problems, for dilatationally-stabilized problems (i.e., essentially for geodetic problems on $SL(n, \mathbb{R})$) and, more generally, for doubly isotropic models when the potential energy is non-trivial but depends only on the deformation invariants, i.e., it has the form $V(q^1, \dots, q^n)$. Let us remind that in this sense quantum mechanics of affine bodies is "simpler" than the classical one where for $n > 2$ there is no simple way of reducing equations of motion to the Cartan subgroup.

It is convenient to start again with the general n , and later on to restrict ourselves to the special cases $n = 2, 3$. Due to the standard orthogonality properties of $\mathcal{D}^{\alpha}_{mn}$, the scalar product of wave functions Ψ may be reduced to one for the amplitudes $f^{\alpha\beta}$ depending only on deformation invariants, i.e.,

$$\langle \Psi_1 | \Psi_2 \rangle = \sum_{\alpha, \beta \in \Omega} \frac{1}{N(\alpha)N(\beta)} \int \sum_{n, m=1}^{N(\alpha)} \sum_{k, l=1}^{N(\beta)} \bar{f}_{1_{nk}}^{\alpha\beta} f_{2_{nl}}^{\alpha\beta} P dq^1 \dots dq^n,$$

where, let us remind, the weight P is given by the following expression:

$$P(q^1, \dots, q^n) = \prod_{i \neq j} |\text{sh}(q^i - q^j)|.$$

If we fix the labels α, β, m, l ("good" quantum numbers for doubly-isotropic problems) and consider the simplified $N(\alpha) \times N(\beta)$ -matrix amplitudes,

$$\Psi^{\alpha\beta}(L; q^1, \dots, q^n; R) = \mathcal{D}^{\alpha}(L) f^{\alpha\beta}(q^1, \dots, q^n) \mathcal{D}^{\beta}(R^{-1}),$$

then the scalar product reduces to

$$\langle \Psi_1^{\alpha\beta} | \Psi_2^{\alpha\beta} \rangle = \frac{1}{N(\alpha)N(\beta)} \int \text{Tr} \left(f_1^{\alpha\beta+} f_2^{\alpha\beta} \right) P dq^1 \dots dq^n,$$

where, obviously, $f_1^{\alpha\beta+}$ denotes the Hermitian conjugate of the matrix $f_1^{\alpha\beta}$.

Obviously, for the general expansion (12) the corresponding formula involves the summation over α, β , and the multiplication of reduced amplitudes and trace operation meant in the sense of two-matrices with the entries labelled by two-indices $f_{nl}^{\alpha\beta}$, i.e.,

$$\langle \Psi_1 | \Psi_2 \rangle = \sum_{\alpha, \beta \in \Omega} \frac{1}{N(\alpha)N(\beta)} \int \text{Tr} \left(f_1^{\alpha\beta+} f_2^{\alpha\beta} \right) P dq^1 \dots dq^n.$$

For the sake of completeness, let us write explicitly

$$\text{Tr} \left(f_1^{\alpha\beta+} f_2^{\alpha\beta} \right) = \sum_{n, m=1}^{N(\alpha)} \sum_{k, l=1}^{N(\beta)} \bar{f}_{1_{nk}}^{\alpha\beta} f_{2_{nl}}^{\alpha\beta}.$$

When we consider the class of problems with α, β, m, l fixed once for all, then one can avoid the divisor $N(\alpha)N(\beta)$, with the proviso of being careful with the normalization of amplitudes so as not to violate the statistical interpretation.

In certain problems it may be convenient to avoid the phase factor P in the above expressions for the scalar product. To achieve this one should introduce rescaled amplitudes given by the matrices $g^{\alpha\beta} := \sqrt{P}f^{\alpha\beta}$. Then the factor P disappears from the above formulas, $f^{\alpha\beta}$ becomes replaced by $g^{\alpha\beta}$, and everything else remains as previously.

20 Metric tensors and arc elements

Essentially everything said above remains valid when discussing the half-integer angular momentum. Orthogonal groups $\text{SO}(n, \mathbb{R})$ in the two-polar decomposition are then replaced by their coverings $\text{Spin}(n)$, but it does not change anything in local analytical expressions. Technically, the only change is that the range of group parameters changes. And where for different parameter values the corresponding elements of $\text{SO}(n, \mathbb{R})$ were identical, in $\text{Spin}(n)$ they are different. It was described above in some details for $\text{SO}(3, \mathbb{R})$ and its covering $\text{Spin}(n) = \text{SU}(2)$, where the main analytical novelty was replacing the range $[0, \pi]$ for the rotation vector magnitude k with $[0, 2\pi]$. All analytical formulas remain formally the same, e.g., those for the generators of left and right regular translations $\mathbf{A}_a, \mathbf{Y}_a$. The metric Killing tensors on $\text{SO}(3, \mathbb{R})$ and $\text{SU}(2)$ normalized to be δ_{ij} in \bar{k} -coordinates at the group identity (thus, differing by the minus one-half factor in comparison with the general Lie-algebraic definition), i.e., $\Gamma(a, b) = -(1/2)\text{Tr}(ab)$ and $\Gamma(a, b) = -2\text{Tr}(ab)$, respectively, on $\text{SO}(3, \mathbb{R})$ and $\text{SU}(2)$; in both cases they are analytically given by the same formula:

$$\Gamma_{ab} = \frac{4}{k^2} \sin^2 \frac{k}{2} \delta_{ab} + \left(1 - \frac{4}{k^2} \sin^2 \frac{k}{2}\right) \frac{k_a k_b}{k^2}.$$

In other words, the corresponding arc element is as follows:

$$ds^2 = \Gamma_{ab} dk^a dk^b = dk^2 + 4 \sin^2 \frac{k}{2} (d\vartheta^2 + \sin^2 \vartheta d\varphi^2).$$

Obviously, this metric is conformally flat, for example, defining new coordinates $\bar{r} = (a/k)\text{tg}(k/4)\bar{k}$, $a > 0$, we obtain that

$$ds^2 = \frac{16a^2}{a^2 + r^2} (dr^2 + r^2 [d\vartheta^2 + \sin^2 \vartheta d\varphi^2]),$$

where the second factor is just the arc element in Euclidean \mathbb{R}^3 expressed in terms of spherical coordinates. This is the conformal mapping of $\text{SU}(2)$ onto \mathbb{R}^3 if we consider the total range $r \in [0, \infty]$. It is interesting that $r \in [0, a]$ on $\text{SO}(3, \mathbb{R})$. This is also some kind of arguments that $\text{SO}(3, \mathbb{R})$ is somehow "imperfect" in comparison with its universal covering $\text{SU}(2)$.

The Haar measure μ in both cases is given by

$$d\mu(\bar{k}) = \frac{4}{\bar{k}^2} \sin^2 \frac{k}{2} d_3 \bar{k} = 4 \sin^2 \frac{k}{2} \sin \vartheta dk d\vartheta d\varphi$$

if we wish its weight function to be equal one in \bar{k} -coordinates at the unit element ($\bar{k} = 0$). But if we wish, as we often do, to normalize the total measure of the compact group to unity, then both cases will differ by a constant factor.

21 Quantizing affine models

One can show after some calculations that the operator $\mathbf{T}_{\text{int}}^{\text{aff}-\text{aff}}$ of kinetic energy invariant under both spatial and material affine transformation is as follows:

$$\mathbf{T}_{\text{int}}^{\text{aff}-\text{aff}} = -\frac{\hbar^2}{2A} \mathbf{D} + \frac{\hbar^2 B}{2A(A+nB)} \frac{\partial^2}{\partial q^2} + \frac{1}{32A} \sum_{a,b} \frac{(\mathbf{M}^a_b)^2}{\text{sh}^2 \frac{q^a - q^b}{2}} - \frac{1}{32A} \sum_{a,b} \frac{(\mathbf{N}^a_b)^2}{\text{ch}^2 \frac{q^a - q^b}{2}}, \quad (21)$$

where A, B are constants as previously in classical formulas, $\mathbf{M}^a_b = -\hat{\mathbf{r}}^a_b - \hat{\mathbf{t}}^a_b$ and $\mathbf{N}^a_b = \hat{\mathbf{r}}^a_b - \hat{\mathbf{t}}^a_b$ (cf. (5), (6)),

$$\mathbf{D} = \frac{1}{P} \sum_a \frac{\partial}{\partial q^a} P \frac{\partial}{\partial q^a} = \sum_a \frac{\partial^2}{\partial (q^a)^2} + \sum_a \frac{\partial \ln P}{\partial q^a} \frac{\partial}{\partial q^a}$$

(every differentiation operator acts on everything on the right of it), P is the previously introduced weight factor.

It is seen that this is almost the previously used classical formula with classical canonical quantities, e.g., $\hat{\rho}^a_b, \hat{\tau}^a_b$ replaced by the corresponding operators $\hat{\mathbf{r}}^a_b, \hat{\mathbf{t}}^a_b$. There is, however, some difference and possibility of an easy mistake in the sector of (q^a, p_a) -variables. Namely, the term involving differentiation with respect to q^a is not, as it might be expected, the usual \mathbb{R}^n -Laplace operator in q^a variables, although it contains such a term. Let us observe that in the $\varphi = LDR^{-1}$ -representation the $\partial/\partial q^a$ operators act only on the $f^{\alpha\beta}$ amplitude, whereas $\hat{\mathbf{r}}^a_b, \hat{\mathbf{t}}^a_b$ act only, respectively, on the L - and R -variables. Therefore, there is no problem of ordering of operators in $\mathbf{T}_{\text{int}}^{\text{aff}-\text{aff}}$. One could get rid off the first derivatives of Ψ with respect to q^a by the substitution which was already used within a slightly different context, namely, $\varphi = \sqrt{P}\Psi$. The action of the last three terms in (21) on φ is exactly as that on Ψ because $\partial/\partial q^a, \mathbf{M}^a_b, \mathbf{N}^a_b$ do not act on $(q^a - q^b)$ -quantities of which P is built; roughly speaking, the \sqrt{P} is "transparent" for these operators. It is no longer the case with the \mathbf{D} -term, both in the good and in the bad senses. Namely, the action of $-(\hbar^2/2A)\mathbf{D}$ on Ψ is represented by the action of the following operator $-(\hbar^2/2A)\tilde{\mathbf{D}}$ on φ :

$$-\frac{\hbar^2}{2A} \tilde{\mathbf{D}} = -\frac{\hbar^2}{2A} \sum_a \frac{\partial^2}{\partial (q^a)^2} + \tilde{\mathbf{V}},$$

where $\tilde{\mathbf{V}}$ is the following artificial potential term:

$$\tilde{\mathbf{V}} = -\frac{\hbar}{2A} \frac{1}{P^2} + \frac{\hbar^2}{4A} \frac{1}{P} \sum_a \left(\frac{\partial P}{\partial q^a} \right)^2.$$

In other words, $\tilde{\mathbf{D}}\varphi = \sqrt{P}\mathbf{D}\Psi$. There are no first derivatives of φ with respect to q^a , and the differential action is given by the usual \mathbb{R}^n -Laplace operator, just as in mechanics of n q^a -particles on \mathbb{R} . But this simplification is only seeming one because, if $n > 2$, it is completely destroyed by the "potential" $\tilde{\mathbf{V}}$. Obviously, in realistic problems concerning deformable objects Hamiltonian should also contain dilatation-stabilizing potential, i.e., $\mathbf{H} = \mathbf{T}_{\text{int}}^{\text{aff-aff}} + \mathbf{V}(q)$. And although such simple $\text{SL}(n, \mathbb{R})$ -geodetic models may successfully describe elastic vibrations, some more general isotropic potentials $V(q^1, \dots, q^n)$ are also acceptable and compatible with the above description.

Quantizing metric-affine and affine-metric kinetic energies we obtain, respectively, the following operators:

$$\begin{aligned} \mathbf{T}_{\text{int}}^{\text{met-aff}} &= -\frac{\hbar^2}{2\alpha} \mathbf{D} - \frac{\hbar^2}{2\beta} \frac{\partial^2}{\partial q^2} + \frac{1}{32\alpha} \sum_{a,b} \frac{(\mathbf{M}^a_b)^2}{\text{sh}^2 \frac{q^a - q^b}{2}} - \frac{1}{32\alpha} \sum_{a,b} \frac{(\mathbf{N}^a_b)^2}{\text{ch}^2 \frac{q^a - q^b}{2}} + \frac{1}{2\mu} \|\mathbf{S}\|^2, \\ \mathbf{T}_{\text{int}}^{\text{aff-met}} &= -\frac{\hbar^2}{2\alpha} \mathbf{D} - \frac{\hbar^2}{2\beta} \frac{\partial^2}{\partial q^2} + \frac{1}{32\alpha} \sum_{a,b} \frac{(\mathbf{M}^a_b)^2}{\text{sh}^2 \frac{q^a - q^b}{2}} - \frac{1}{32\alpha} \sum_{a,b} \frac{(\mathbf{N}^a_b)^2}{\text{ch}^2 \frac{q^a - q^b}{2}} + \frac{1}{2\mu} \|\mathbf{V}\|^2, \end{aligned}$$

with the same meaning of operator symbols as above and the same relationship between inertial constants (α, β, μ) and the primary ones (I, A, B) as above.

22 Potential case

As mentioned, for Hamiltonians $\mathbf{H} = \mathbf{T} + \mathbf{V}$ with dilatation-stabilizing potentials $V(q)$, or more generally, with doubly-isotropic potentials $V(q^1, \dots, q^n)$, the action of operators \mathbf{M}^a_b and \mathbf{N}^a_b become algebraic and standard, and the stationary Schrödinger equation, i.e., energy eigenproblem $\mathbf{H}\Psi = E\Psi$, splits into family of eigenproblems for the amplitudes $f^{\alpha\beta}$; they are partial differential equations involving q^a -variables only:

$$\mathbf{H}^{\alpha\beta} f^{\alpha\beta} = E^{\alpha\beta} f^{\alpha\beta},$$

where $f^{\alpha\beta}$ for any $\alpha, \beta \in \Omega$ is an $N(\alpha) \times N(\beta)$ matrix depending on q^1, \dots, q^n . In a consequence of the double (spatial and material) isotropy, this problem is $N(\alpha) \times N(\beta)$ -fold degenerate, i.e., for every component of $f^{\alpha\beta}$ there exists an $N(\alpha) \times N(\beta)$ -dimensional subspace of solutions. Let us remind that in the primary symbols $f_{nk}^{\alpha\beta}$ the indices m, l just label the degeneracy of solutions for every $f_{nk}^{\alpha\beta}$. $\mathbf{H}^{\alpha\beta}$ is an $N(\alpha) \times N(\beta)$ -matrix of second-order differential operators, $\mathbf{H}^{\alpha\beta} = \mathbf{T}^{\alpha\beta} + \mathbf{V}$, where \mathbf{V} denotes a dilatation-stabilizing or general doubly-isotropic potential, and $\mathbf{T}^{\alpha\beta}$ denotes the kinetic energy operator. It is one of

the previous ones restricted to the corresponding (α, β) -subspace. Therefore, for the affine-affine, metric-affine, and affine-metric models we have, respectively,

$$\mathbf{T}^{\alpha\beta} f^{\alpha\beta} = -\frac{\hbar^2}{2A} \mathbf{D} f^{\alpha\beta} + \frac{\hbar^2 B}{2A(A+nB)} \frac{\partial^2}{\partial q^2} f^{\alpha\beta} \quad (22)$$

$$+ \frac{1}{32A} \sum_{a,b} \frac{(\overleftarrow{S}^{\beta a}_b - \overrightarrow{S}^{\alpha a}_b)^2}{\text{sh}^2 \frac{q^a - q^b}{2}} f^{\alpha\beta} - \frac{1}{32A} \sum_{a,b} \frac{(\overleftarrow{S}^{\beta a}_b + \overrightarrow{S}^{\alpha a}_b)^2}{\text{ch}^2 \frac{q^a - q^b}{2}} f^{\alpha\beta},$$

$$\mathbf{T}^{\alpha\beta} f^{\alpha\beta} = -\frac{\hbar^2}{2\alpha} \mathbf{D} f^{\alpha\beta} - \frac{\hbar^2}{2\beta} \frac{\partial^2}{\partial q^2} f^{\alpha\beta} - \frac{\hbar^2}{2\mu} C(\alpha, 2) f^{\alpha\beta} \quad (23)$$

$$+ \frac{1}{32\alpha} \sum_{a,b} \frac{(\overleftarrow{S}^{\beta a}_b - \overrightarrow{S}^{\alpha a}_b)^2}{\text{sh}^2 \frac{q^a - q^b}{2}} f^{\alpha\beta} - \frac{1}{32\alpha} \sum_{a,b} \frac{(\overleftarrow{S}^{\beta a}_b + \overrightarrow{S}^{\alpha a}_b)^2}{\text{ch}^2 \frac{q^a - q^b}{2}} f^{\alpha\beta},$$

$$\mathbf{T}^{\alpha\beta} f^{\alpha\beta} = -\frac{\hbar^2}{2\alpha} \mathbf{D} f^{\alpha\beta} - \frac{\hbar^2}{2\beta} \frac{\partial^2}{\partial q^2} f^{\alpha\beta} - \frac{\hbar^2}{2\mu} C(\beta, 2) f^{\alpha\beta} \quad (24)$$

$$+ \frac{1}{32\alpha} \sum_{a,b} \frac{(\overleftarrow{S}^{\beta a}_b - \overrightarrow{S}^{\alpha a}_b)^2}{\text{sh}^2 \frac{q^a - q^b}{2}} f^{\alpha\beta} - \frac{1}{32\alpha} \sum_{a,b} \frac{(\overleftarrow{S}^{\beta a}_b + \overrightarrow{S}^{\alpha a}_b)^2}{\text{ch}^2 \frac{q^a - q^b}{2}} f^{\alpha\beta},$$

where the meaning of Casimir eigenvalues $C(\alpha, 2)$, $C(\beta, 2)$ like in (17). The constants α , β , μ are exactly as previously; do not confuse them with labels α , β at $f^{\alpha\beta}$. In the physical case $n = 3$, $\alpha = s = 0, 1/2, 1, \dots \in \mathbb{N}/2 \cup \{0\}$ and similarly $\beta = j = 0, 1/2, 1, \dots \in \mathbb{N}/2 \cup \{0\}$ assuming that the half-integer values of angular momentum and vorticity are admitted. Otherwise we would have $s, j \in \mathbb{N} \cup \{0\}$. Obviously, in this case $C(s, 2) = -s(s+1)$, $C(j, 2) = -j(j+1)$, and the additional constants in the last two formulas are simply $(\hbar^2/2\mu)s(s+1)$, $(\hbar^2/2\mu)j(j+1)$, expressions close to the heart of any physicist. Let us stress that, even if half-integers are admitted, there is a restriction that $(j-s)$ must be integer, i.e., j and s have the same "half-ness". In any case, it must be so if wave functions are to be well-defined on $\overline{\text{GL}}(3, \mathbb{R})$ not only on the "artificial" configuration space $\text{SU}(2) \times \mathbb{R}^3 \times \text{SU}(2)$. If they are to be statistically interpretable in $\text{GL}(3, \mathbb{R})$ itself, then only the terms with half-integer (s, j) or integer (s, j) may be separately superposed, no mutual superposition admissible (although some blasphemous doubts may be raised against this superselection, i.e., against statistical interpretation in $\text{GL}(3, \mathbb{R})$).

In three-dimensional case the above-mentioned additional terms

$$(\hbar^2/2\mu)s(s+1), \quad (\hbar^2/2\mu)j(j+1)$$

seem to be physically interesting and, at least qualitatively, compatible with some experimental data. It is so as if the doubly affine background (affine invariance in space and in the body) was responsible for some fundamental part of the spectra, which later on, the more the μ is smaller, splits due to some internal rotations. The term $(\hbar^2/2\mu)s(s+1)$ is physically intuitive and classically corresponds to the situation when in the system some regime of rigid rotations

was established after time of transition processes. But, perhaps, $(\hbar^2/2\mu)j(j+1)$ appearing in the affine-metrical model is even more interesting. Being a formal analogue of certain aspects of angular momentum, it is not angular momentum and may be perhaps semiclassically related to the isotopic spin or similar internal quantities ruled by $SU(2)$ and appearing in nuclear and elementary particle physics.

Remark: Just as previously, the terms with the first-order derivatives of $f^{\alpha\beta}$ with respect to q^a may be avoided by the substitution

$$g^{\alpha\beta} := \sqrt{P} f^{\alpha\beta},$$

which was also used for simplifying the scalar product. But then again the artificial potential \mathbf{V} appears in all reduced Schrödinger equations.

By the way, one can have both things, i.e., $(\hbar^2/2\mu)s(s+1)$ and $(\hbar^2/2\mu)j(j+1)$ terms. For this purpose we would have to use the kinetic energy consisting of four terms:

$$T_{\text{int}} = \frac{I_1}{2} g_{ik} g^{jl} \Omega_j^i \Omega_l^k + \frac{I_2}{2} \eta_{KL} \eta^{MN} \hat{\Omega}_M^K \hat{\Omega}_N^L + \frac{A}{2} \hat{\Omega}_L^K \hat{\Omega}_K^L + \frac{B}{2} \hat{\Omega}_K^K \hat{\Omega}_L^L,$$

where the last two terms might be as well written as $(A/2)\Omega_j^i \Omega_i^j + (B/2)\Omega_i^i \Omega_j^j$. In matrix language, using Cartesian coordinates $g_{ik} =_* \delta_{ik}$, $\eta_{AB} =_* \delta_{AB}$, we would simply write that

$$\begin{aligned} T_{\text{int}} &= \frac{I_1}{2} \text{Tr}(\Omega^T \Omega) + \frac{I_2}{2} \text{Tr}(\hat{\Omega}^T \hat{\Omega}) + \frac{A}{2} \text{Tr}(\hat{\Omega}^2) + \frac{B}{2} (\text{Tr} \hat{\Omega})^2 \\ &= \frac{I_1}{2} \text{Tr}(\Omega^T \Omega) + \frac{I_2}{2} \text{Tr}(\hat{\Omega}^T \hat{\Omega}) + \frac{A}{2} \text{Tr}(\Omega^2) + \frac{B}{2} (\text{Tr} \Omega)^2. \end{aligned} \quad (25)$$

But now some reproach might be raised that, doing as above, we forget our primary motivation concerning the dynamical $GL(n, \mathbb{R})$ -invariance and return to models which are only orthogonally invariant (geometrically speaking, $O(V, g)$ - and $O(U, \eta)$ -invariant), and it is again only pure kinematics that is ruled by affine group. This would be true, and we indeed do not insist on the above model. Let us notice, however, that this model, having still high dynamical symmetry, may also work as a purely geodetic model encoding a kind of elastic bounded vibrations without any extra introduced potential. Moreover, due to the lack of dilatational invariance, it is not excluded (we are not yet sure; this is a conjecture) that even dilatation-stabilizing potentials would not be necessary.

23 Doubly-isotropic d'Alembert models

The above remarks about models (25) again put our attention on the doubly isotropic "d'Alembert" models of classical kinetic energy ([20]2.1), i.e., ([20]4.21) with the factorization $\mathcal{A}^{K_i L_j} = I g_{ij} \eta^{KL}$. The corresponding kinetic part of the classical kinetic Hamiltonian $\mathcal{T}_{\text{int}}^{\text{d.A.}}$ was given by ([20]6.68) with the same meaning of M^a_b , N^a_b as above, $Q^a = D_{aa}$ are diagonal elements of D , and P_a

are canonical momenta conjugate to Q^a . This time, as a measure particularly convenient for quantization, the usual Lebesgue measure l on $L(n)$ should be used, $dl(\varphi) = d\varphi^1_1 \cdots d\varphi^n_n$. In terms of the two-polar splitting, $dl(L, D, R) = P_l(Q)d\mu(L)d\mu(R)dQ^1 \cdots dQ^n$, where μ , as previously, is the Haar measure on $SO(n, \mathbb{R})$, and the weight factor P_l is now given by the following expression:

$$P_l = \prod_{a \neq b} |(Q^a)^2 - (Q^b)^2| = \prod_{a \neq b} |(Q^a + Q^b)(Q^a - Q^b)|.$$

Everything concerning quantization looks in a similar way like previously for affinely-invariant models. For example, expansion of wave functions Ψ with respect to $\mathcal{D}^\alpha(L)$, $\mathcal{D}^\beta(R)$ with $f^{\alpha\beta}(D)$ -reduced amplitudes is exactly the same. The difference appears in details concerning the integration procedure, just the weight factor P_l is substituted instead of P . Also, in spite of formal similarities, the particular form of the kinetic energy operator is different,

$$\mathbf{T}_{\text{int}}^{\text{d.A}} = -\frac{\hbar^2}{2I} \mathbf{D}_l + \frac{1}{8I} \sum_{a,b} \frac{(\mathbf{M}^a_b)^2}{(Q^a - Q^b)^2} + \frac{1}{8I} \sum_{a,b} \frac{(\mathbf{N}^a_b)^2}{(Q^a + Q^b)^2},$$

where now

$$\mathbf{D}_l = \frac{1}{P_l} \sum_a \frac{\partial}{\partial Q^a} P_l \frac{\partial}{\partial Q^a} = \sum_a \frac{\partial^2}{\partial (Q^a)^2} + \sum_a \frac{\partial \ln P_l}{\partial Q^a} \frac{\partial}{\partial Q^a}.$$

Just as previously, the weight factor P_l in the scalar product and first-order differentiations $\partial/\partial Q^a$ may be avoided by rescaling $\varphi = \sqrt{P_l} \Psi$, but in the resulting differential operator acting on φ also some rather unpleasant potential term appears, i.e.,

$$\tilde{\mathbf{V}}_l = -\frac{\hbar}{2I} \frac{1}{P_l^2} + \frac{\hbar^2}{4I} \frac{1}{P_l} \sum_a \left(\frac{\partial P_l}{\partial Q^a} \right)^2.$$

It is obvious that without an appropriate potential term \mathbf{V} the geodetic Hamiltonian $\mathbf{T}^{\text{d.A}}$ cannot work in theory of deformable objects because just as on the classical level it describes only purely scattering, non-bounded motions. Indeed, the above operator

$$\mathbf{T}^{\text{d.A}} = -\frac{\hbar^2}{2I} \Delta^{n^2} = -\frac{\hbar^2}{2I} \sum_{i,A} \frac{\partial^2}{\partial (\varphi^i_A)^2}$$

is simply proportional to the usual Laplace operator in \mathbb{R}^{n^2} written in non-typical coordinates.

Therefore, the only realistic applications of the above \mathbf{T} are those as a term of some doubly isotropic Hamiltonian $\mathbf{H} = \mathbf{T}^{\text{d.A}} + \mathbf{V}(Q^1, \dots, Q^n)$. Just as previously, due to the double isotropy of the model, the resulting stationary

Schrödinger equation $\mathbf{H}\Psi = E\Psi$ splits into the family of equations for partial amplitudes $f^{\alpha\beta}$ depending only on q^a -variables, $\mathbf{H}^{\alpha\beta} f^{\alpha\beta} = E^{\alpha\beta} f^{\alpha\beta}$, where

$$\begin{aligned} \mathbf{H}^{\alpha\beta} f^{\alpha\beta} = & -\frac{\hbar^2}{2I} \mathbf{D}_l f^{\alpha\beta} + \frac{1}{8I} \sum_{a,b} \frac{(\overleftarrow{S}^{\beta a}_b - \overrightarrow{S}^{\alpha a}_b)^2}{(Q^a - Q^b)^2} f^{\alpha\beta} \\ & + \frac{1}{8I} \sum_{a,b} \frac{(\overleftarrow{S}^{\beta a}_b + \overrightarrow{S}^{\alpha a}_b)^2}{(Q^a + Q^b)^2} f^{\alpha\beta} + V(Q^1, \dots, Q^n) f^{\alpha\beta}. \end{aligned} \quad (26)$$

For d'Alembert models, the problem of coverings and multi-valued wave functions looks exactly like in affine theories. Simply $\mathrm{SO}(n, \mathbb{R})$ -groups in the two-polar decomposition must be replaced by the coverings $\mathrm{Spin}(n)$. In particular, for $n = 3$ when $\alpha, \beta = s, j = 0, 1/2, 1, \dots$, everything said above remains true, and S^{sa}_b, S^{ja}_b are replaced by the standard Wigner matrices of angular momentum, S^s_a, S^j_a .

24 Usual Wigner matrices of angular momentum

In three dimensions those terms of the affine-affine reduced operator $\mathbf{T}^{\alpha\beta}$ (22) which contain the factor $1/32A$ may be written in the following form involving the usual Wigner matrices S^j_a :

$$\begin{aligned} \sum_{a=1}^3 \left[\frac{(S^s_a)^2 f^{sj} - 2S^s_a f^{sj} S^j_a + f^{sj} (S^j_a)^2}{16A s \hbar^2 \frac{q^b - q^c}{2}} \right. \\ \left. - \frac{(S^s_a)^2 f^{sj} + 2S^s_a f^{sj} S^j_a + f^{sj} (S^j_a)^2}{16A c \hbar^2 \frac{q^b - q^c}{2}} \right], \end{aligned} \quad (27)$$

where in any a -th term of both summations we have obviously $b \neq a, c \neq a, b \neq c$ (it is clear that it does not matter what is the sequence of b, c).

The same holds for the metric-affine and affine-metric models (23), (24), with the proviso that the inertial factor A is replaced by α . As mentioned, the last constant-multiplicator terms are respectively $(\hbar^2/2\mu)s(s+1)f^{sj}$ and $(\hbar^2/2\mu)j(j+1)f^{sj}$. Similarly, in reduced d'Alembert expressions (26) the terms with the $1/8I$ -factor become for $n = 3$:

$$\begin{aligned} \sum_{a=1}^3 \left[\frac{(S^s_a)^2 f^{sj} - 2S^s_a f^{sj} S^j_a + f^{sj} (S^j_a)^2}{4I(Q^b - Q^c)^2} \right. \\ \left. + \frac{(S^s_a)^2 f^{sj} + 2S^s_a f^{sj} S^j_a + f^{sj} (S^j_a)^2}{4I(Q^b + Q^c)^2} \right] \end{aligned} \quad (28)$$

with the same as previously convention concerning indices a, b, c .

For affinely-invariant geodetic models the bounded state L^2 -solutions appear for particular relationships between s and j (α and β) in n dimensions). For the d'Alembert models of kinetic energy this is impossible, an appropriate potential $V(Q^1, \dots, Q^n)$ must be always used.

Both the affine and d'Alembert expressions (27), (28) become particularly simple for the lowest possible values of rotational quantum numbers s, j , and then there exists some hope for rigorous or at least numerical solutions. Thus, for $s = j = 0$ the corresponding expressions vanish at all, and the resulting Schrödinger equations for f^{00} are purely scalar. For $s = j = 1/2$ we obtain the spinor-spinor state, which is also relatively simple because then $S^{1/2}_a = (\hbar/2)\sigma_a$, $(S^{1/2}_a)^2 = (\hbar^2/4)I_2$, where, obviously, σ_a are Pauli matrices, and I_2 is the unit 2×2 matrix.

25 Two-dimensional case on the classical level

In some physical problems also the two-dimensional case $n = 2$ may be physically interesting [12]. And in any case it is mathematically exceptionally simple. This is, so to speak, "pathological" simplicity following from the commutativity of $SO(2, \mathbb{R})$. Although this exceptional simplicity is rather "exotic" from the point of view of the general n , it may suggest some guiding hints for analysis of this general situation.

The main two-dimensional peculiarity is that

$$\hat{\rho} = \rho = S, \quad \hat{\tau} = \tau = -V.$$

This is exactly due to the commutativity of $SO(2, \mathbb{R})$. Because of this, the convenient quantities $\hat{\rho}, \hat{\tau}$ are constants of motion for geodetic models and models with doubly-invariant potentials. It was not the case for $n > 2$, where only S, V are constants of motion (for invariant geodetic models and, more generally, for doubly-isotropic models). But it is just the use of $\hat{\rho}$ and $\hat{\tau}$, or equivalently M and N , that simplifies the problem and enables one to perform a partial separation of variables, especially effective on the quantum level. If $n = 2$, the two things coincide, and the problem may be effectively reduced to the Cartan subgroup of diagonal matrices (deformation invariants) even on the classical level.

Let us begin with the classical description. In the two-polar decomposition $\varphi = LDR^{-1}$ we shall use the following parameterization:

$$L = \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix}, \quad R = \begin{bmatrix} \cos \beta & -\sin \beta \\ \sin \beta & \cos \beta \end{bmatrix}, \quad D = \begin{bmatrix} \exp q^1 & 0 \\ 0 & \exp q^2 \end{bmatrix}.$$

The splitting $GL^+(2, \mathbb{R}) = \mathbb{R}^+ SL(2, \mathbb{R})$ is well-suited to coordinates

$$q = (q^1 + q^2)/2, \quad x = q^2 - q^1,$$

and their conjugate canonical momenta, respectively,

$$p = p_1 + p_2, \quad p_x = (p_2 - p_1)/2.$$

Before using these convenient coordinates, let us express classical kinetic energies in terms of primary variables. First of all, let us notice the obvious fact that the angular velocities of L - and R -rotators are given, respectively, by

$$\chi = \frac{dL}{dt}L^{-1} = L^{-1}\frac{dL}{dt} = \hat{\chi} = \frac{d\alpha}{dt} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

$$\vartheta = \frac{dR}{dt}R^{-1} = R^{-1}\frac{dR}{dt} = \hat{\vartheta} = \frac{d\beta}{dt} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

The corresponding spin and vorticity quantities are given (in canonical representation) by the following expressions:

$$S = \rho = \hat{\rho} = p_\alpha \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad V = -\tau = -\hat{\tau} = p_\beta \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

where p_α , p_β are, respectively, canonical momenta conjugate to α , β . The corresponding duality pairings are as follows:

$$p_\alpha \frac{d\alpha}{dt} = \frac{1}{2}\text{Tr}(S\chi) = \frac{1}{2}\text{Tr}(\rho\chi) = \frac{1}{2}\text{Tr}(\hat{\rho}\hat{\chi}),$$

$$p_\beta \frac{d\beta}{dt} = \frac{1}{2}\text{Tr}(V\vartheta) = -\frac{1}{2}\text{Tr}(\tau\vartheta) = -\frac{1}{2}\text{Tr}(\hat{\tau}\hat{\vartheta}),$$

where $d\alpha/dt$, $d\beta/dt$ are arbitrary virtual velocities of the variables α , β .

The corresponding classical quantities $M = -\hat{\rho} - \hat{\tau}$, $N = \hat{\rho} - \hat{\tau}$ are, respectively, given by the following expressions:

$$M = \mathbf{m} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad N = \mathbf{n} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

where $\mathbf{m} := p_\beta - p_\alpha$, $\mathbf{n} := p_\beta + p_\alpha$ may be interpreted as canonical momenta conjugate to the corresponding "mixtures" of angles β , α : $\gamma := (\beta - \alpha)/2$, $\delta := (\beta + \alpha)/2$, i.e., $\alpha = \delta - \gamma$, $\beta = \delta + \gamma$. In fact, one can easily show that $\mathbf{m}\dot{\gamma} + \mathbf{n}\dot{\delta} = p_\alpha\dot{\alpha} + p_\beta\dot{\beta}$ for arbitrary virtual velocities occurring in these formulas, thus, $\mathbf{m} = p_\gamma = p_\beta - p_\alpha$, $\mathbf{n} = p_\delta = p_\beta + p_\alpha$, and conversely, $p_\alpha = (\mathbf{n} - \mathbf{m})/2$, $p_\beta = (\mathbf{n} + \mathbf{m})/2$. The previously used magnitudes of S , V become:

$$\|S\| = |p_\alpha| = \frac{1}{2}|\mathbf{n} - \mathbf{m}|, \quad \|V\| = |p_\beta| = \frac{1}{2}|\mathbf{n} + \mathbf{m}|.$$

For the classical affine-affine kinetic energy ([20]6.69) in Hamiltonian representation we obtain the following expression:

$$\mathcal{T}_{\text{int}}^{\text{aff-aff}} = \frac{1}{2A} (p_1^2 + p_2^2) - \frac{B}{2A(A+2B)} p^2 + \frac{1}{16A} \frac{\mathbf{m}^2}{\text{sh}^2 \frac{q^2 - q^1}{2}} - \frac{1}{16A} \frac{\mathbf{n}^2}{\text{ch}^2 \frac{q^2 - q^1}{2}};$$

meaning of symbols A , B is like previously, and $n = 2$ is substituted to constant factors.

Similarly, for the metrical-affine and affine-metrical models we obtain, respectively,

$$\begin{aligned}\mathcal{T}_{\text{int}}^{\text{met-aff}} &= \frac{1}{2\alpha}(p_1^2 + p_2^2) + \frac{1}{2\beta}p^2 + \frac{1}{16\alpha}\frac{\mathbf{m}^2}{\text{sh}^2\frac{q^2-q^1}{2}} - \frac{1}{16\alpha}\frac{\mathbf{n}^2}{\text{ch}^2\frac{q^2-q^1}{2}} + \frac{1}{8\mu}(\mathbf{n} - \mathbf{m})^2, \\ \mathcal{T}_{\text{int}}^{\text{aff-met}} &= \frac{1}{2\alpha}(p_1^2 + p_2^2) + \frac{1}{2\beta}p^2 + \frac{1}{16\alpha}\frac{\mathbf{m}^2}{\text{sh}^2\frac{q^2-q^1}{2}} - \frac{1}{16\alpha}\frac{\mathbf{n}^2}{\text{ch}^2\frac{q^2-q^1}{2}} + \frac{1}{8\mu}(\mathbf{n} + \mathbf{m})^2,\end{aligned}$$

where meaning of constants α, β, μ is like previously, but with $n = 2$ substituted, thus, $\alpha = I + A$, $\beta = -(I + A)(I + A + 2B)/B$, $\mu = (I^2 - A^2)/I$. As \mathbf{m} and \mathbf{n} , or equivalently p_α and p_β , are now constants of motion, it is seen that for geodesic problems and for problems with doubly-isotropic potentials $V(q^1, q^2)$, e.g., with dilatation-stabilizing ones $V(q)$, everything reduces to the two-dimensional dynamics in variables q^1, q^2 ruled by the effective Hamiltonian obtained by the formal substitution of fixed values p_α, p_β (or \mathbf{m}, \mathbf{n}) to the above expressions. Moreover, for $\text{SL}(2, \mathbb{R})$ -geodesic problems, or for $\text{GL}(2, \mathbb{R})$ -problems with separated variables potentials $V(q, x) = V_{\text{dil}}(q) + V_{\text{sh}}(x)$, everything reduces trivially to independent one-dimensional motions. In the above geodesic models it is only the relationship between constant values of \mathbf{m}, \mathbf{n} that decides whether the motion is oscillatory or unbounded. The first case happens, obviously, when $|\mathbf{n}| > |\mathbf{m}|$; then at large "distances" $|q^2 - q^1|$ the attractive ch^{-2} -term prevails. On the contrary, if $|\mathbf{n}| < |\mathbf{m}|$, one deals with the repulsive case, i.e., with the decaying motion of invariants q^1, q^2 . This is the simplest example of the fact mentioned above that affinely-invariant geodesic models admit an open family of bounded (vibrating) and an open family of non-bounded (decaying) motions. Obviously, for general $n > 2$ the situation is more complicated because then M^a_b, N^a_b fail to be constants of motion and perform oscillations somehow coupled with those of q^a . Using new variables q, x, p, p_x , we can rewrite the above models of \mathcal{T} in the following forms:

$$\begin{aligned}\mathcal{T}_{\text{int}}^{\text{aff-aff}} &= \frac{p^2}{4(A + 2B)} + \frac{p_x^2}{A} + \frac{(p_\alpha - p_\beta)^2}{16A\text{sh}^2\frac{x}{2}} - \frac{(p_\alpha + p_\beta)^2}{16A\text{ch}^2\frac{x}{2}}, \\ \mathcal{T}_{\text{int}}^{\text{met-aff}} &= \frac{p^2}{4(I + A + 2B)} + \frac{p_x^2}{I + A} + \frac{Ip_\alpha^2}{I^2 - A^2} \\ &\quad + \frac{(p_\alpha - p_\beta)^2}{16(I + A)\text{sh}^2\frac{x}{2}} - \frac{(p_\alpha + p_\beta)^2}{16(I + A)\text{ch}^2\frac{x}{2}}, \\ \mathcal{T}_{\text{int}}^{\text{aff-met}} &= \frac{p^2}{4(I + A + 2B)} + \frac{p_x^2}{I + A} + \frac{Ip_\beta^2}{I^2 - A^2} \\ &\quad + \frac{(p_\alpha - p_\beta)^2}{16(I + A)\text{sh}^2\frac{x}{2}} - \frac{(p_\alpha + p_\beta)^2}{16(I + A)\text{ch}^2\frac{x}{2}}.\end{aligned}$$

In the special case $n = 2$, it is easily seen that on the level of variables q, x all these geodesic models have identical dynamics. The difference appears only on the level of angular variables α, β . And, just as for the general n , the same is

true if we introduce to Hamiltonians some doubly-isotropic potentials $V(q, x)$. In particular, this is true for dilatation-stabilizing potentials $V(q)$, i.e., in a sense, for geodetic invariant models on $\text{SL}(2, \mathbb{R})$ (incompressible bodies).

26 Quantization of two-dimensional models

Let us now turn to quantization. The Haar measure λ on $\text{GL}(2, \mathbb{R})$ is given by the following expression: $d\lambda(\alpha; q^1, q^2; \beta) = |\text{sh}(q^1 - q^2)| d\alpha d\beta dq^1 dq^2$, i.e., $d\lambda(\alpha; q, x; \beta) = |\text{sh}x| d\alpha d\beta dq dx$, $P = |\text{sh}x|$. The Peter-Weyl expansion with respect to the L, R -factors of the two-polar splitting is just the usual double Fourier series:

$$\Psi(\alpha; q, x; \beta) = \sum_{m, n \in \mathbb{Z}} f^{mn}(q, x) e^{im\alpha} e^{in\beta}.$$

The reduced kinetic Hamiltonian corresponding to $\mathcal{T}_{\text{int}}^{\text{aff}-\text{aff}}$ is as follows:

$$\begin{aligned} \mathbf{T}^{mn} f^{mn} = & -\frac{\hbar^2}{A} \mathbf{D}_x f^{mn} - \frac{\hbar^2}{4(A+2B)} \frac{\partial^2 f^{mn}}{\partial q^2} \\ & + \frac{\hbar^2(n-m)^2}{16A \text{sh}^2 \frac{x}{2}} f^{mn} - \frac{\hbar^2(n+m)^2}{16A \text{ch}^2 \frac{x}{2}} f^{mn}, \end{aligned}$$

where

$$\mathbf{D}_x f^{mn} = \frac{1}{|\text{sh}x|} \frac{\partial}{\partial x} \left(|\text{sh}x| \frac{\partial f^{mn}}{\partial x} \right).$$

For the metric-affine and affine-metric models $\mathcal{T}_{\text{int}}^{\text{met}-\text{aff}}$, $\mathcal{T}_{\text{int}}^{\text{aff}-\text{met}}$ we obtain, respectively, the following expressions:

$$\begin{aligned} \mathbf{T}^{mn} f^{mn} = & -\frac{\hbar^2}{I+A} \mathbf{D}_x f^{mn} - \frac{\hbar^2}{4(I+A+2B)} \frac{\partial^2 f^{mn}}{\partial q^2} \\ & + \frac{\hbar^2(n-m)^2}{16(I+A) \text{sh}^2 \frac{x}{2}} f^{mn} - \frac{\hbar^2(n+m)^2}{16(I+A) \text{ch}^2 \frac{x}{2}} f^{mn} + \frac{I\hbar^2 m^2}{I^2 - A^2} f^{mn}, \\ \mathbf{T}^{mn} f^{mn} = & -\frac{\hbar^2}{I+A} \mathbf{D}_x f^{mn} - \frac{\hbar^2}{4(I+A+2B)} \frac{\partial^2 f^{mn}}{\partial q^2} \\ & + \frac{\hbar^2(n-m)^2}{16(I+A) \text{sh}^2 \frac{x}{2}} f^{mn} - \frac{\hbar^2(n+m)^2}{16(I+A) \text{ch}^2 \frac{x}{2}} f^{mn} + \frac{I\hbar^2 n^2}{I^2 - A^2} f^{mn}, \end{aligned}$$

It is seen that in all these expressions the complete separation between dilatational and incompressible motion is very effectively described in analytical terms just due to the use of coordinates q, x . Obviously, for geodetic Hamiltonians on $\text{GL}(2, \mathbb{R})$ the energy spectrum is continuous (and classical trajectories are unbounded; in a sense equivalent facts) because dilatational motion is free. As in the general case, this fact is physically avoided by introducing to the Hamiltonian some dilatation-stabilizing potential $V_{\text{dil}}(q)$. On the quantum level the simplest possible model is the potential well.

This is, in a sense, reduction to the geodetic quantum problem on $SL(2, \mathbb{R})$. Obviously, the problem with $V_{\text{dil}}(q)$ remains explicitly separable. It remains so also for a more general class of doubly isotropic potentials, e.g., for ones explicitly splitting, $V(q, x) = V_{\text{dil}}(q) + V_{\text{sh}}(x)$, but perhaps also for more general ones. Solutions of the corresponding stationary Schrödinger equations may be sought in the following form: $f^{mn}(q, x) = \varphi^{mn}(q)\chi^{mn}(x)$; the problem reduces then to one-dimensional Schrödinger equations for φ^{mn} and χ^{mn} . And now, in the special two-dimensional case, it is explicitly seen that there exists a discrete spectrum (bounded situations) for χ -functions, i.e., for the isochoric $SL(2, \mathbb{R})$ -problem, even in the purely geodetic case without any potential $V_x(x)$. And this is true in spite of the non-compactness of the $SL(2, \mathbb{R})$ -configuration space. Everything depends on the mutual relationship between "rotational" quantum numbers m, n . If $|n + m| > |n - m|$, the attractive χ^{-2} -term prevails at large "distances" $|x| \rightarrow \infty$ and the spectrum is discrete. In the opposite case, if $|n + m| < |n - m|$, it is continuous.

For the affine-affine geodetic model on $SL(2, \mathbb{R})$, the total spectrum (total in the sense of solutions for all possible $m, n \in \mathbb{Z}$) is not bounded from below; this might seem undesirable. For the metric-affine and affine-metric geodetic problems on $SL(2, \mathbb{R})$, the spectrum may be bounded from below (and so is the corresponding kinetic energy). Everything depends on the mutual relationship between inertial constants I, A, B , which play the role of some controlling parameters.

27 Usual two-dimensional d'Alembert models

For comparison, let us quote a few corresponding formulas for the "usual" d'Alembert model in two dimensions. We restrict ourselves to the doubly-isotropic model. The classical kinetic Hamiltonian may be expressed as follows:

$$\mathcal{T}_{\text{int}}^{\text{d,A}} = \frac{1}{2I} (P_1^2 + P_2^2) + \frac{1}{4I} \frac{\mathfrak{m}^2}{(Q^1 - Q^2)^2} + \frac{1}{4I} \frac{\mathfrak{n}^2}{(Q^1 + Q^2)^2},$$

with the same meaning of symbols as previously. Let us stress that Q^a are diagonal elements of D , and now the variables $q^a = \ln Q^a$ would be completely useless. The quantity P_l is given simply by the following expression:

$$P_l = \left| (Q^1)^2 - (Q^2)^2 \right| = \left| (Q^1 + Q^2) (Q^1 - Q^2) \right|,$$

and the usual Lebesgue measure on $L(2, \mathbb{R}) \simeq \mathbb{R}^4$ is expressed as follows:

$$dl(\alpha; Q^1, Q^2; \beta) = P_l(Q^1, Q^2) d\alpha d\beta dQ^1 dQ^2.$$

As mentioned, geodetic models are non-physical (and, by the way, the above coordinates would be completely artificial for them). There is, however, a class of physically reasonable doubly isotropic potentials $V(Q^1, Q^2)$ for which the corresponding Hamiltonians $H = \mathcal{T} + V$ describe integrable systems admitting

solutions in terms of separation of variables. This fact is obvious when, instead of Q^1, Q^2 , the $(\pi/4)$ -rotated coordinates Q^+, Q^- on the plane of deformation invariants are used, $Q^\pm := (Q^1 \pm Q^2)/\sqrt{2}$. The polar and elliptic coordinates on the (Q^+, Q^-) -plane are also convenient, i.e., $Q^+ = r \cos \varphi$, $Q^- = r \sin \varphi$ and $Q^+ = \text{ch} \rho \cos \lambda$, $Q^- = \text{sh} \rho \sin \lambda$.

There exist physically reasonable potentials V for which the corresponding Hamiltonian problems are separable (thus, obviously, integrable) in coordinates (Q^+, Q^-) , (r, φ) , or (ρ, λ) . There are also interesting superintegrable (degenerate) models separable simultaneously in two or even three of the above coordinate systems.

On the quantized level the reduced Schrödinger equation has the following form: $\mathbf{H}^{mn} f^{mn} = E^{mn} f^{mn}$, where

$$\begin{aligned} \mathbf{H}^{mn} f^{mn} &= \mathbf{T}^{mn} f^{mn} + \mathbf{V}(Q^1, Q^2) f^{mn} \\ &= -\frac{\hbar^2}{2I} \mathbf{D}_I f^{mn} + \frac{\hbar^2 m^2}{4I(Q^1 - Q^2)^2} f^{mn} + \frac{\hbar^2 n^2}{4I(Q^1 + Q^2)^2} f^{mn} + \mathbf{V}(Q^1, Q^2) f^{mn}. \end{aligned}$$

Obviously,

$$\mathbf{D}_I f = \frac{1}{P_I} \sum_{a=1}^2 \frac{\partial}{\partial Q^a} \left(P_I \frac{\partial f}{\partial Q^a} \right).$$

Everything said above about separability of the classical problems remains true on the quantized level. Again the coordinate systems (Q^+, Q^-) , (r, φ) , (ρ, λ) are crucial.

28 Hamiltonian systems on $\mathbf{U}(n)$

To finish these quantization remarks let us mention briefly about Hamiltonian systems on $\mathbf{U}(n)$, i.e., in a sense, affine systems with "compactified deformation invariants" ([20]6.69). The resulting kinetic energy operator has the following form:

$$\mathbf{T} = -\frac{\hbar^2}{2A} \mathbf{D}_U + \frac{\hbar^2 B}{2A(A + nB)} \frac{\partial^2}{\partial q^2} + \frac{1}{32A} \sum_{a,b} \frac{(\mathbf{M}^a_b)^2}{\sin^2 \frac{q^a - q^b}{2}} + \frac{1}{32A} \sum_{a,b} \frac{(\mathbf{N}^a_b)^2}{\cos^2 \frac{q^a - q^b}{2}},$$

where

$$\begin{aligned} \mathbf{D}_U &= \frac{1}{P_U} \sum_a \frac{\partial}{\partial q^a} P_U \frac{\partial}{\partial q^a} = \sum_a \frac{\partial^2}{\partial (q^a)^2} + \sum_a \frac{\partial \ln P_U}{\partial q^a} \frac{\partial}{\partial q^a}, \\ P_U &= \prod_{a \neq b} |\sin(q^a - q^b)|. \end{aligned}$$

The Haar measure is given by the expression

$$d\lambda_U(L, D, R) = P_U d\mu(L) d\mu(R) dq^1 \cdots dq^n,$$

where μ , as previously, denotes the Haar measure on $\text{SO}(n, \mathbb{R})$.

Obviously, $U(n)$ is compact, thus, all classical trajectories for geodetic models are bounded and the corresponding quantum spectrum is discrete. Nevertheless, more general models with doubly-isotropic potentials, i.e., $\mathbf{H} = \mathbf{T} + \mathbf{V}(q^1, \dots, q^n)$, may be also of physical interest.

The problem splits again, just as in the $GL(n, \mathbb{R})$ -case, into the family of reduced problems resulting from the Fourier analysis on $SO(n, \mathbb{R})$ performed both in the L - and R -variables: $\mathbf{H}^{\alpha\beta} f^{\alpha\beta} = E^{\alpha\beta} f^{\alpha\beta}$, where

$$\begin{aligned} \mathbf{H}^{\alpha\beta} f^{\alpha\beta} = & -\frac{\hbar^2}{2I} \mathbf{D}_U f^{\alpha\beta} - \frac{\hbar^2}{2\beta} \frac{\partial^2 f^{\alpha\beta}}{\partial q^2} + \frac{1}{32A} \sum_{a,b} \frac{\left(\overleftarrow{S}^{\beta a_b} - \overrightarrow{S}^{\alpha a_b}\right)^2}{\sin^2 \frac{q^a - q^b}{2}} f^{\alpha\beta} \\ & + \frac{1}{32A} \sum_{a,b} \frac{\left(\overleftarrow{S}^{\beta a_b} + \overrightarrow{S}^{\alpha a_b}\right)^2}{\cos^2 \frac{q^a - q^b}{2}} f^{\alpha\beta} + \mathbf{V}(q^1, \dots, q^n) f^{\alpha\beta}. \end{aligned}$$

Just as in the $GL(n, \mathbb{R})$ -models, particularly simple are physical dimensions $n = 2, 3$. The former one has also certain very peculiar features and admits simple calculations based on integrable models and separability techniques. Namely, \mathbf{H}^{mn} acts as follows:

$$\begin{aligned} \mathbf{H}^{mn} f^{mn} = & -\frac{\hbar^2}{A} \mathbf{D}_x f^{mn} + \frac{\hbar^2(n-m)^2}{16A \sin^2 \frac{x}{2}} f^{mn} + \frac{\hbar^2(n+m)^2}{16A \cos^2 \frac{x}{2}} f^{mn} + \mathbf{V}_x(x) f^{mn} \\ & - \frac{\hbar^2}{4(A+2B)} \frac{\partial^2 f^{mn}}{\partial q^2} + \mathbf{V}_q(q) f^{mn}, \end{aligned}$$

where the Haar measure has the expression $d\lambda_U(\alpha; q, x; \beta) = |\sin x| d\alpha d\beta dq dx$ and

$$\mathbf{D}_x f = \frac{1}{|\sin x|} \frac{\partial}{\partial x} \left(|\sin x| \frac{\partial f}{\partial x} \right).$$

The problem also separates, in particular, for geodetic problems, $V = 0$, or for potentials of the above-mentioned form $V(q, x) = V_{\text{dil}}(q) + V_{\text{sh}}(x)$.

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